ETH Zürich

Chair of Software Engineering

Software Verification

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Lecture 8: Abstract Interpretation

Plan for today's lecture

- In the first part we discuss program slicing as another example of an application of data flow analysis.

- In the second part we discuss abstract interpretation, a general framework for expressing program analyses.

Program Slicing

Program slicing

```
1     sum := 0
2     prod := 1
3     i := 0
4     while i < y do
5         sum := sum + x
6         prod := prod * x
7         i := i + 1
      end
8     print(sum)
9     print(prod)
```

```
1     sum := 0

3     i := 0
4     while i < y do
5         sum := sum + x

7         i := i + 1
      end
8     print(sum)
```

"What program statements potentially affect the value of variable sum at line 8 of the program?"

Program slicing

- Program slicing provides an answer to the question

> "What program statements potentially affect the values of the variables at program point l?"

- The resulting program statements are called the program slice.
- The program point l is called the slicing criterion.
- An observer focusing on the slicing criterion (i.e. only observing values of the variables at program point l) cannot distinguish a run of the program from the run of its slice.

Applications of program slicing

- **Debugging:** Slicing lets the programmer focus on the program part relevant to a certain failure, which might lead to quicker detection of a fault.
- **Testing:** Slicing can minimize test cases, i.e. find the smallest set of statements that produces a certain failure (good for regression testing).
- **Parallelization:** Slicing can determine parts of the program which can be computed independently of each other and can thus be parallelized.

Classification

- Static slicing vs. dynamic slicing
  - Static: general, not considering a particular input
  - Dynamic: computed for a fixed input, therefore smaller slices can be obtained
- Backward slicing vs. forward slicing
  - Backward: "Which statements affect the execution of a statement?"
  - Forward: "Which statements are affected by the execution of a certain statement?"
- In the following we present an algorithm for static backward slicing.

Program slice

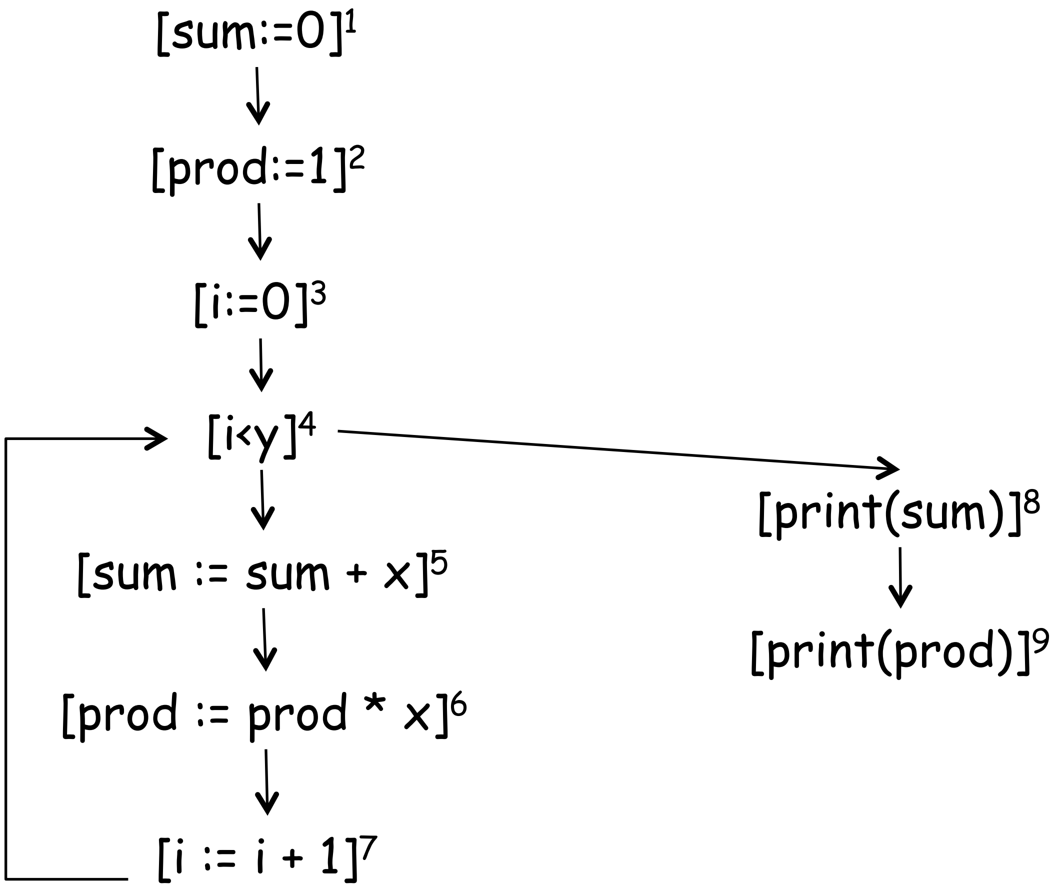
A backward slice S of program P with respect to slicing criterion l is any executable program with the following properties:

1. S can be obtained by deleting zero or more statements from P.
2. If P halts on input I, then the values of the variables at program point l are the same in P and in S every time program point l is executed.

Slicing algorithm

- We present a slicing algorithm for static backward slicing.
- Many different approaches, we show one that constructs a program dependence graph (PDG).
- A PDG is a directed graph with two types of edges:
  - Data dependencies: given by data-flow analysis
  - Control dependencies: program point l is control-dependent on program point l' if
    (1) l' labels the guard of a control structure
    (2) the execution of l depends on the outcome of the evaluation of the guard at l'

Control flow graph of the example program



[sum:=0]^1

[prod:=1]^2

[i:=0]^3

[i<y]^4

[sum := sum + x]^5

[prod := prod * x]^6

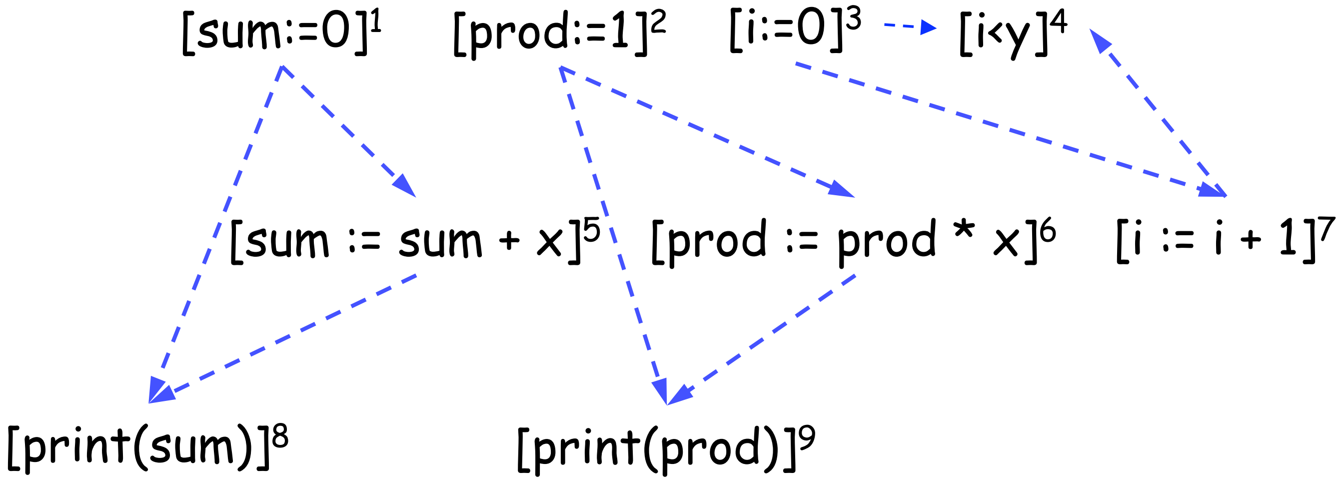
[i := i + 1]^7

[print(sum)]^8

[print(prod)]^9

Example: Program dependence graph

1. Data dependence subgraph



$[sum{:=}0]^1$ $[prod{:=}1]^2$ $[i{:=}0]^3$ $[i<y]^4$

$[sum := sum + x]^5$ $[prod := prod * x]^6$ $[i := i + 1]^7$

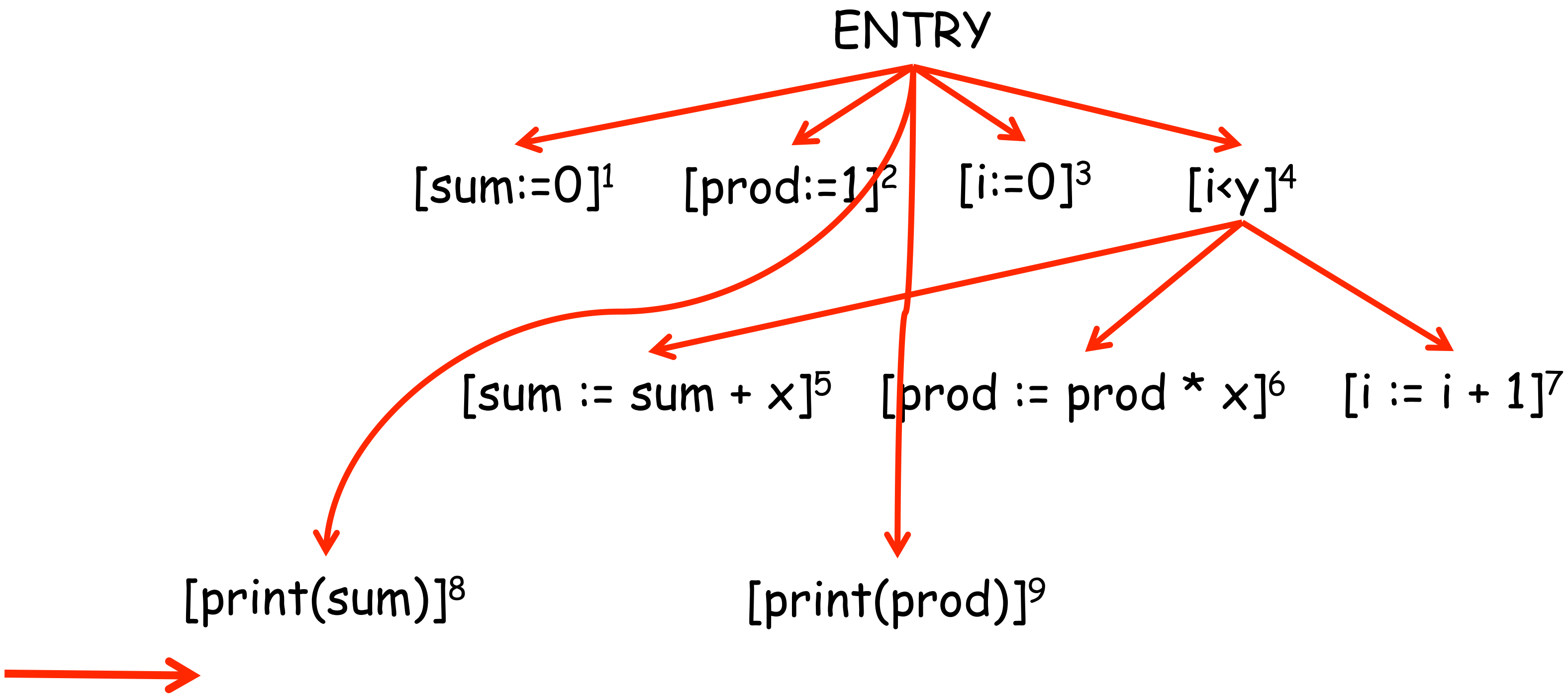
$[print(sum)]^8$ $[print(prod)]^9$

$$\dashrightarrow \{(l, l') \mid l \in \bigcup_{\substack{x \text{ used} \\ \text{in block } l'}} UD(x, l') \text{ where } l' \text{ labels a block}\}$$

(self-loops are omitted)

Example: Program dependence graph

2. Control dependence subgraph



ENTRY

$[sum:=0]^1$ $[prod:=1]^2$ $[i:=0]^3$ $[i<y]^4$

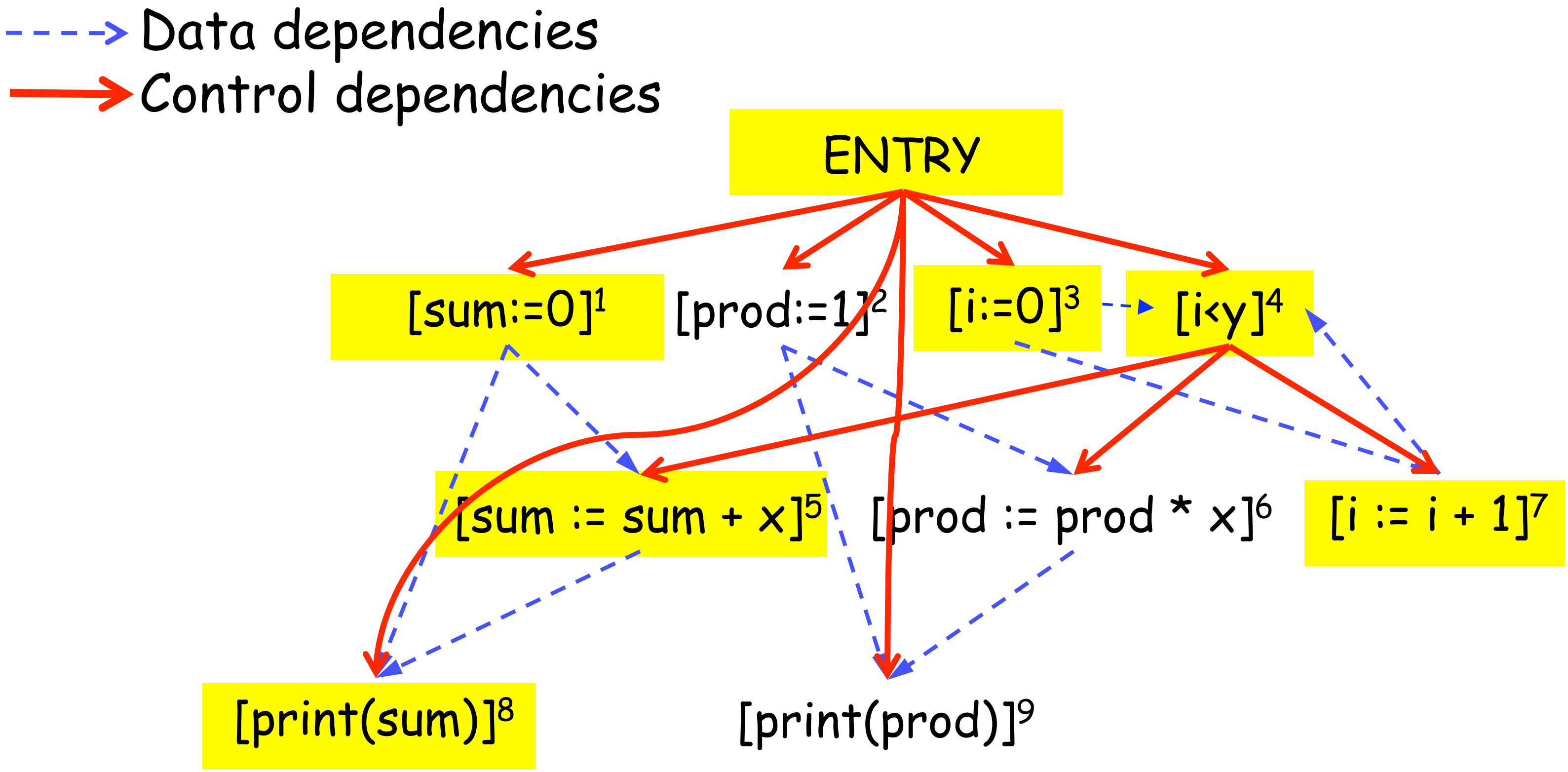
$[sum := sum + x]^5$ $[prod := prod * x]^6$ $[i := i + 1]^7$

$[print(sum)]^8$ $[print(prod)]^9$

(1) Edge from special node ENTRY to any node not within any control structure (such as while, if-then-else)

(2) Edge from any guard of a control structure to any statement within the control structure

Example: Computing the program slice



Slicing using the PDG:

(1) Take as initial node the one given by the slicing criterion

(2) Include all nodes which the initial node transitively depends upon (use both data- and control-dependencies)

Abstract Interpretation

Introduction

One framework to rule them all

- In the past lecture we have introduced a particular style of program analysis: data flow analysis.

- For these types of analyses, and others, a main concern is correctness: how do we know that a particular analysis produces sound results (does not forget possible errors)?

- In the following we discuss abstract interpretation, a general framework for describing program analyses and reasoning about their correctness.

Main ideas: Concrete computations

- An ordinary program describes computations in some concrete domain of values.
  - **Example:** program states that record the integer value of every program variable.

$$\sigma \in State = Var \rightarrow Z$$

- Possible computations can be described by the concrete semantics of the programming language used.

Main ideas: Abstract computations

- Abstract interpretation of a program describes computation in a different, abstract domain.
  - **Example:** program states that only record a specific property of integers, instead of their value: their sign, whether they are even/odd, or contained in [-32768, 32767] etc.

$$\sigma \in \text{AbstractState} = \text{Var} \rightarrow \{\text{even}, \text{odd}\}$$

- In order to obtain abstract computations, an abstract semantics for the programming language has to be defined.
- Abstract interpretation provides a framework for proving that the abstract semantics is sound with respect to the concrete semantics.

The collecting semantics

We assume the state of a program to be modeled as:

$$\sigma \in State = Var \to Z$$

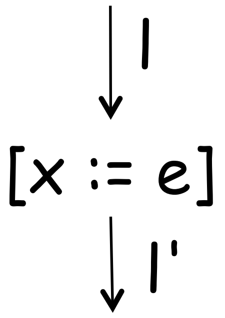
We will use the following notation for function update:

$$\sigma[x \mapsto k](y) = \begin{cases} k & \text{if } x = y \\ \sigma(y) & \text{otherwise} \end{cases}$$

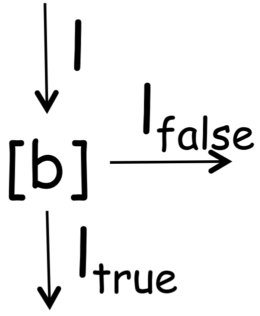
We construct the collecting semantics as a function which gives for every program label the set of all possible states.

$$C : Labels \to \wp(State)$$

Rules of the collecting semantics

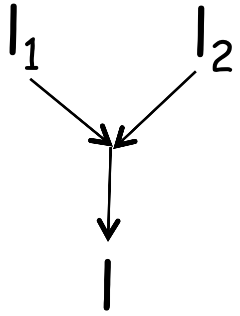


$$C_{l'} = \{\sigma[x \mapsto n] \mid \sigma \in C_l \text{ and } C[e]\sigma = n\}$$



$$C_{ltrue} = \{\sigma \mid \sigma \in C_l \text{ and } C[b]\sigma = \text{true}\}$$

$$C_{lfalse} = \{\sigma \mid \sigma \in C_l \text{ and } C[b]\sigma = \text{false}\}$$

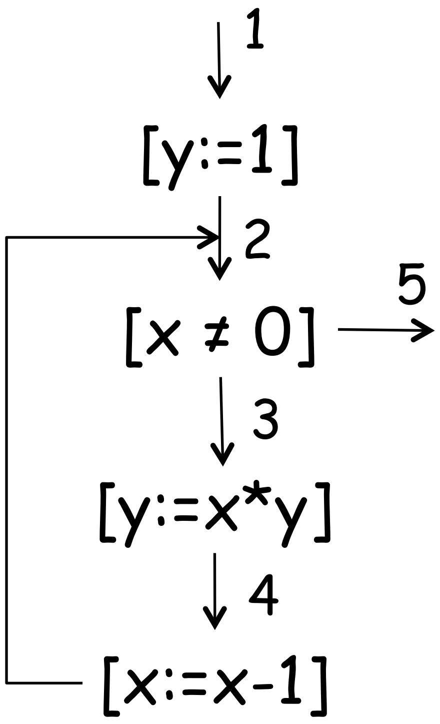


$$C_l = C_{l1} \cup C_{l2}$$

Note: In difference to the lecture on program analysis, labels are not on blocks, but on edges.

Example: Collecting semantics

Assume x > 0.



$C_1 = \{\sigma \mid \sigma(x) > 0\}$

$$\begin{aligned} C_2 = {} & \{\sigma[y \mapsto 1] \mid \sigma \in C_1\} \cup \\ & \{\sigma[x \mapsto \sigma(x) - 1] \mid \sigma \in C_4\} \end{aligned}$$

$C_3 = C_2 \cap \{\sigma \mid \sigma(x) \neq 0\}$

$C_4 = \{\sigma[y \mapsto \sigma(x) \cdot \sigma(y)] \mid \sigma \in C_3\}$

$C_5 = C_2 \cap \{\sigma \mid \sigma(x) = 0\}$

Solving the equations

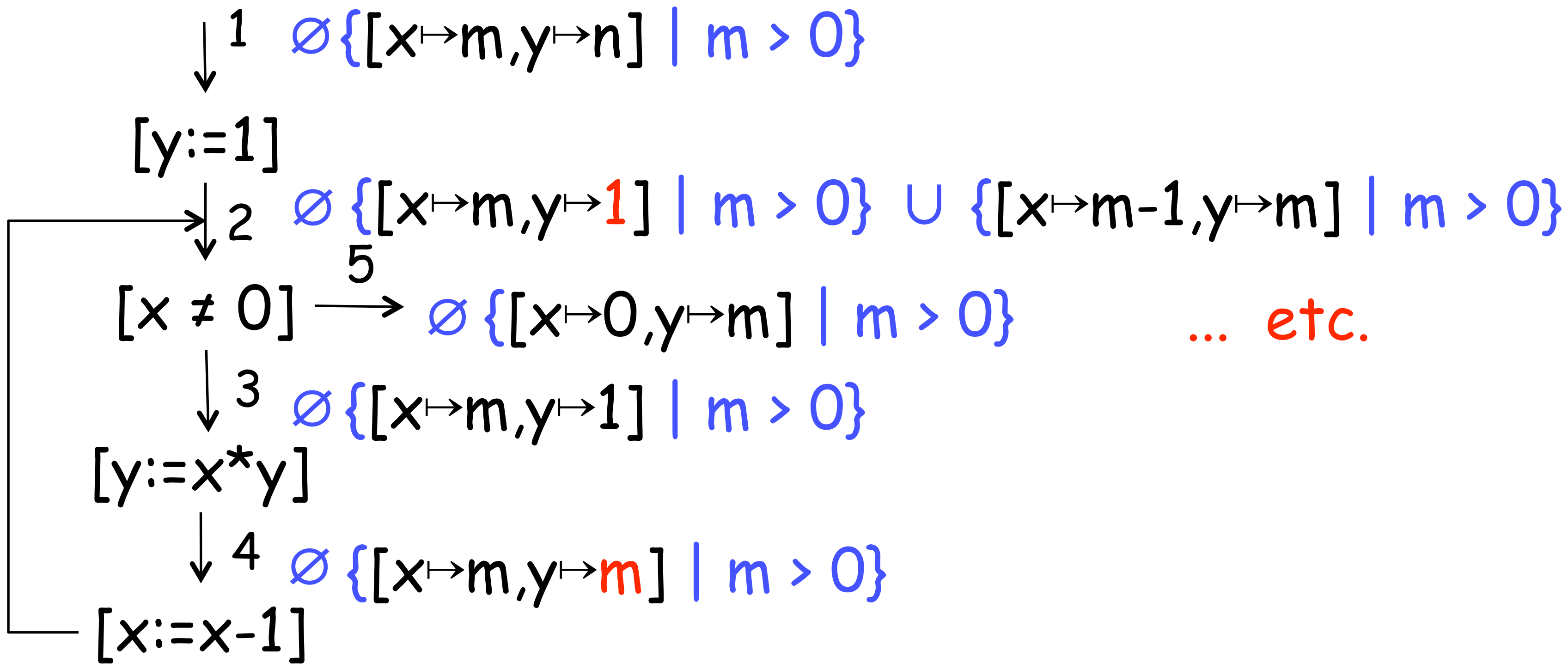
- The equation system we obtain has variables $C_1, \ldots, C_5$ which are interpreted over the complete lattice $\wp(\mathrm{State})$.
- We can express the equation system as a monotone function $F : \wp(\mathrm{State})^5 \to \wp(\mathrm{State})^5$

$$F(C_1, \ldots, C_5) = (\{\sigma \mid \sigma(x) > 0\}, \ldots, C_2 \cap \{\sigma \mid \sigma(x) = 0\})$$

- Using Tarski's Fixed Point Theorem, we know that a least fixed point exists.
- We have seen: The least fixed point can be computed by repeatedly applying F, starting with the bottom element $\bot = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)$ of the complete lattice until stabilization.

$$F(\bot) \sqsubseteq F(F(\bot)) \sqsubseteq \ldots \sqsubseteq F^n(\bot) = F^{n+1}(\bot)$$

Example: Fixed Point Computation



$\downarrow 1$ $\varnothing\{[x\mapsto m, y\mapsto n] \mid m > 0\}$

[y:=1]

$\downarrow 2$ $\varnothing\{[x\mapsto m, y\mapsto 1] \mid m > 0\} \cup \{[x\mapsto m-1, y\mapsto m] \mid m > 0\}$

[x ≠ 0] $\xrightarrow{5}$ $\varnothing\{[x\mapsto 0, y\mapsto m] \mid m > 0\}$ … etc.

$\downarrow 3$ $\varnothing\{[x\mapsto m, y\mapsto 1] \mid m > 0\}$

[y:=x*y]

$\downarrow 4$ $\varnothing\{[x\mapsto m, y\mapsto m] \mid m > 0\}$

[x:=x-1]

$$C_1 = \{\sigma \mid \sigma(x) > 0\}$$

$$C_2 = \{\sigma[y \mapsto 1] \mid \sigma \in C_1\} \cup \{\sigma[x \mapsto \sigma(x) - 1] \mid \sigma \in C_4\}$$

$$C_3 = C_2 \cap \{\sigma \mid \sigma(x) \neq 0\}$$

$$C_4 = \{\sigma[y \mapsto \sigma(x)\cdot\sigma(y)] \mid \sigma \in C_3\}$$

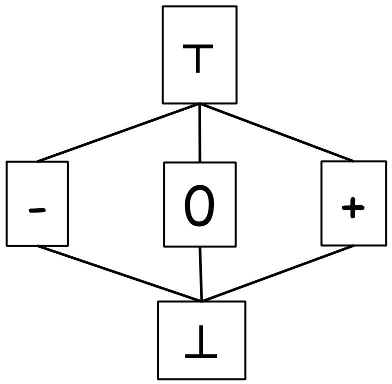
$$C_5 = C_2 \cap \{\sigma \mid \sigma(x) = 0\}$$

Domain for Sign Analysis

We want to focus on the sign of integers, using the domain

$$\sigma \in AbstractState = Var \to Signs$$

where Signs is the following structure:



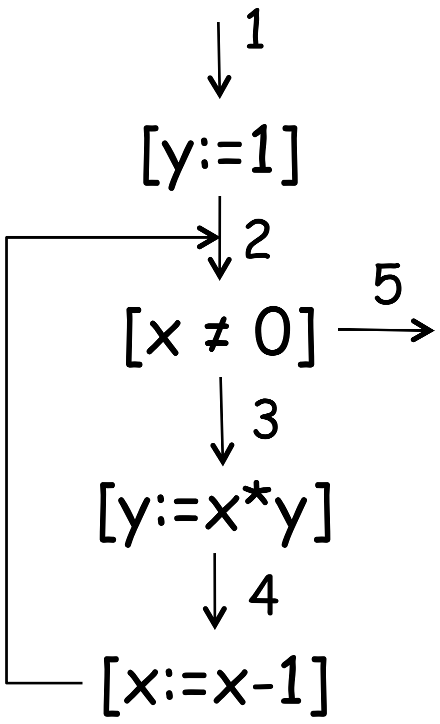
⊤	represents all integers
+	the positive integers
-	the negative integers
0	the set {0}
⊥	the empty set

How is such a structure called?

A complete lattice

Example: Sign Analysis

Assume x > 0. Use the abstract domain for sign analysis.



$$A_1 = [x \mapsto +, y \mapsto \top]$$

$$A_2 = A_1[y \mapsto +] \sqcup A_4[x \mapsto A_4(x) \ominus +]$$

$$A_3 = A_2$$

$$A_4 = A_3[y \mapsto A_3(x) \otimes A_3(y)]$$

$$A_5 = A_2 \sqcap [x \mapsto 0, y \mapsto \top]$$

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Abstract Interpretation

Foundations

Introductory example: Expressions

A little language of expressions

Syntax

$e ::= n \mid e * e$

Concrete semantics

$C[n] = n$

$C[e * e] = C[e] \cdot C[e]$

Example

$C[-3 * 2 * -5] = C[-3 * 2] \cdot C[-5] = C[-3 * 2] \cdot (-5) = \ldots = 30$

Introductory example: Abstraction

Assume that we are not interested in the value of an expression but only in its sign:

- Negative: –
- Zero: 0
- Positive: +

Abstract semantics

$A[n] = \mathrm{sign}(n)$

$A[e * e] = A[e] \otimes A[e]$

$\otimes$	-	0	+
-	+	0	-
0		0	0
+			+

Example

$A[-3 * 2 * -5] = A[-3 * 2] \otimes A[-5] = A[-3 * 2] \otimes (-) = \ldots =$

$= (-) \otimes (+) \otimes (-) = (+)$

Introductory example: Soundness

- We want to express that the abstract semantics correctly describes the sign of a corresponding concrete computation.
- For this we first link each concrete value to an abstract value:

Representation function

$\beta : Z \rightarrow \{-, 0, +\}$

$$\beta(n) = \begin{cases} - & \text{if } n < 0 \\ 0 & \text{if } n = 0 \\ + & \text{if } n > 0 \end{cases}$$

Introductory example: Soundness

- Conversely, we can also link abstract values to the set of concrete values they describe:

Concretization function

$\gamma : \{-, 0, +\} \rightarrow \wp(\mathbb{Z})$

$$\gamma(s) = \begin{cases} \{n \mid n < 0\} & \text{if } s = - \\ \{0\} & \text{if } s = 0 \\ \{n \mid n > 0\} & \text{if } s = + \end{cases}$$

- Soundness then describes intuitively that the concrete value of an expression is described by its abstract value:

$$\forall e.\ C[e] \in \gamma(A[e])$$

Extending the language

Syntax

$e ::= n \mid e * e \mid e + e \mid -e$

Abstract semantics

$A[n] = \text{sign}(n)$

$A[-e] = \ominus A[e]$

$A[e + e] = A[e] \oplus A[e]$

	-	0	+
⊖	+	0	-

⊕	-	0	+
-	-	-	?
0		0	+
+			+

Observation: The abstract domain {-,0,+} is not closed under the interpretation of addition.

Extending the abstract domain

We have to introduce an additional abstract value:

⊤ "top" – (any value)

⊕	-	0	+	⊤
-	-	-	⊤	⊤
0		0	+	⊤
+			+	⊤
⊤				⊤

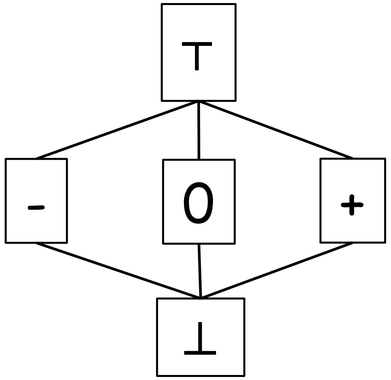
The new abstract domain

We can extend the concretization function to the new abstract domain {-,0,+, ⊤, ⊥} (add ⊥ for completeness):

$\gamma(\top) = \mathbf{Z}$ $\gamma(\bot) = \emptyset$

We obtain the following structure when drawing the partial order induced by

$a \leq b \text{ iff } \gamma(a) \subseteq \gamma(b)$



How is such a structure called?

A complete lattice

Construction of complete lattices

- If we know some complete lattices, we can construct new ones by combining them
- Such constructions become important when designing new analyses with complex analysis domains

Example: Total function space

Let $(D_1, \sqsubseteq_1)$ be a partially ordered set and let S be a set. Then $(D, \sqsubseteq)$, defined as follows, is a complete lattice:

- $D = S \rightarrow D_1$ ("space of total functions")
- $f \sqsubseteq f'$ iff $\forall\ s \in S : f(s) \sqsubseteq_1 f'(s)$ ("point-wise ordering")

The framework of abstract interpretation

- Starting from a concrete domain C, define an abstract domain $(A, \sqsubseteq)$, which must be a complete lattice
- Define a representation function β that maps a concrete value to its best abstract value

$$\beta : C \rightarrow A$$

- From this we can derive the concretization function γ

$$\gamma : A \rightarrow \wp(C)$$

$$\gamma(a) = \{c \in C \mid \beta(c) \sqsubseteq a\}$$

and abstraction function α for sets of concrete values

$$\alpha : \wp(C) \rightarrow A$$

$$\alpha(C) = \sqcup \{\beta(c) \mid c \in C\}$$

Galois connections

- The following properties of α and γ hold:

Monotonicity

(1) α and γ are monotone functions

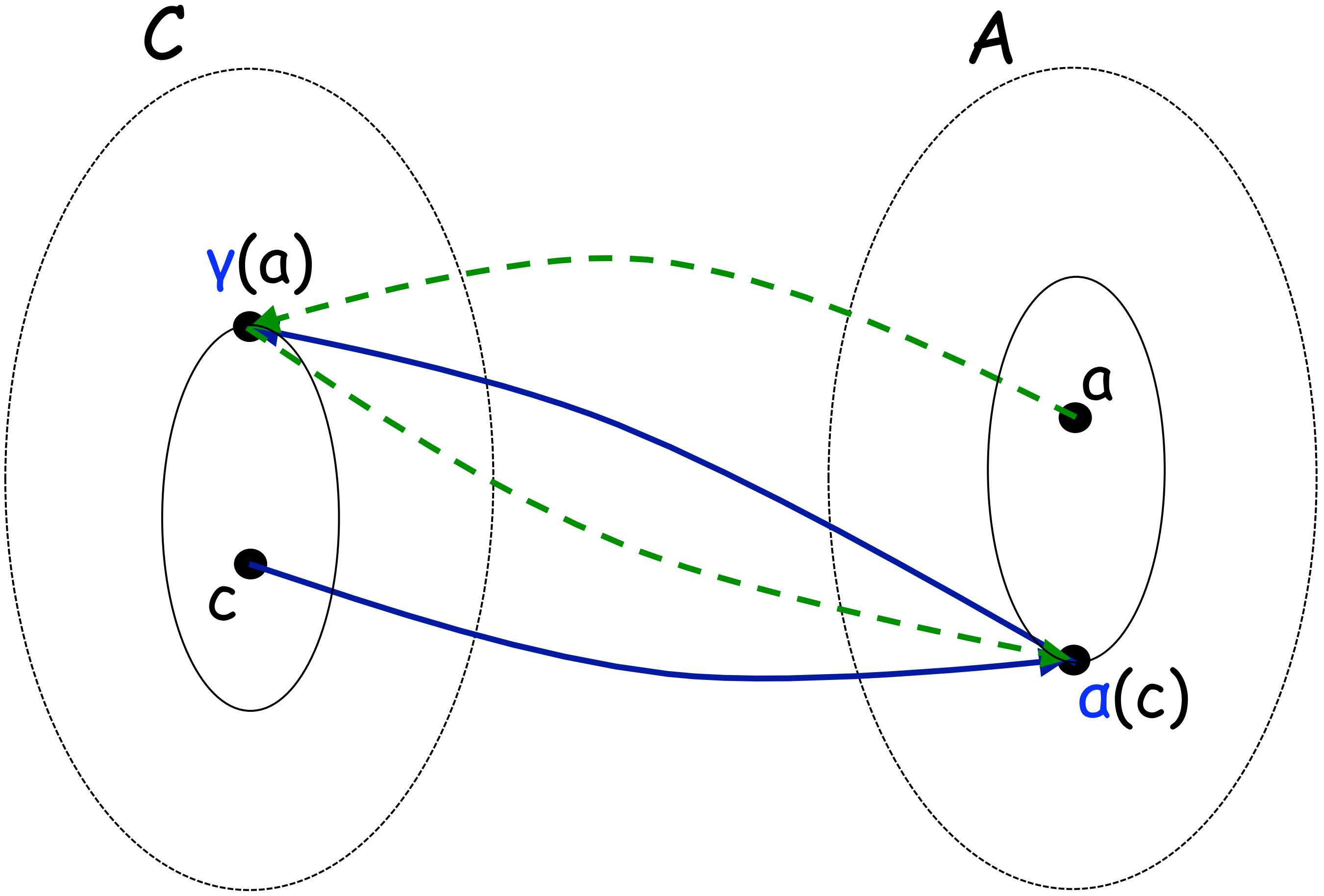
Galois connection

(3) $c \subseteq \gamma(\alpha(c))$ for all $c \in \wp(C)$

(2) $a \sqsupseteq \alpha(\gamma(a))$ for all $a \in A$

- Galois connection: This property means intuitively that the functions α and γ are "almost inverses" of each other.

Figure: Galois connection



Galois insertions

- For a Galois connection, there may be several elements of A that describe the same element in C
- As a result, A may contain elements which are irrelevant for describing C
- The concept of Galois insertion fixes this:

Monotonicity

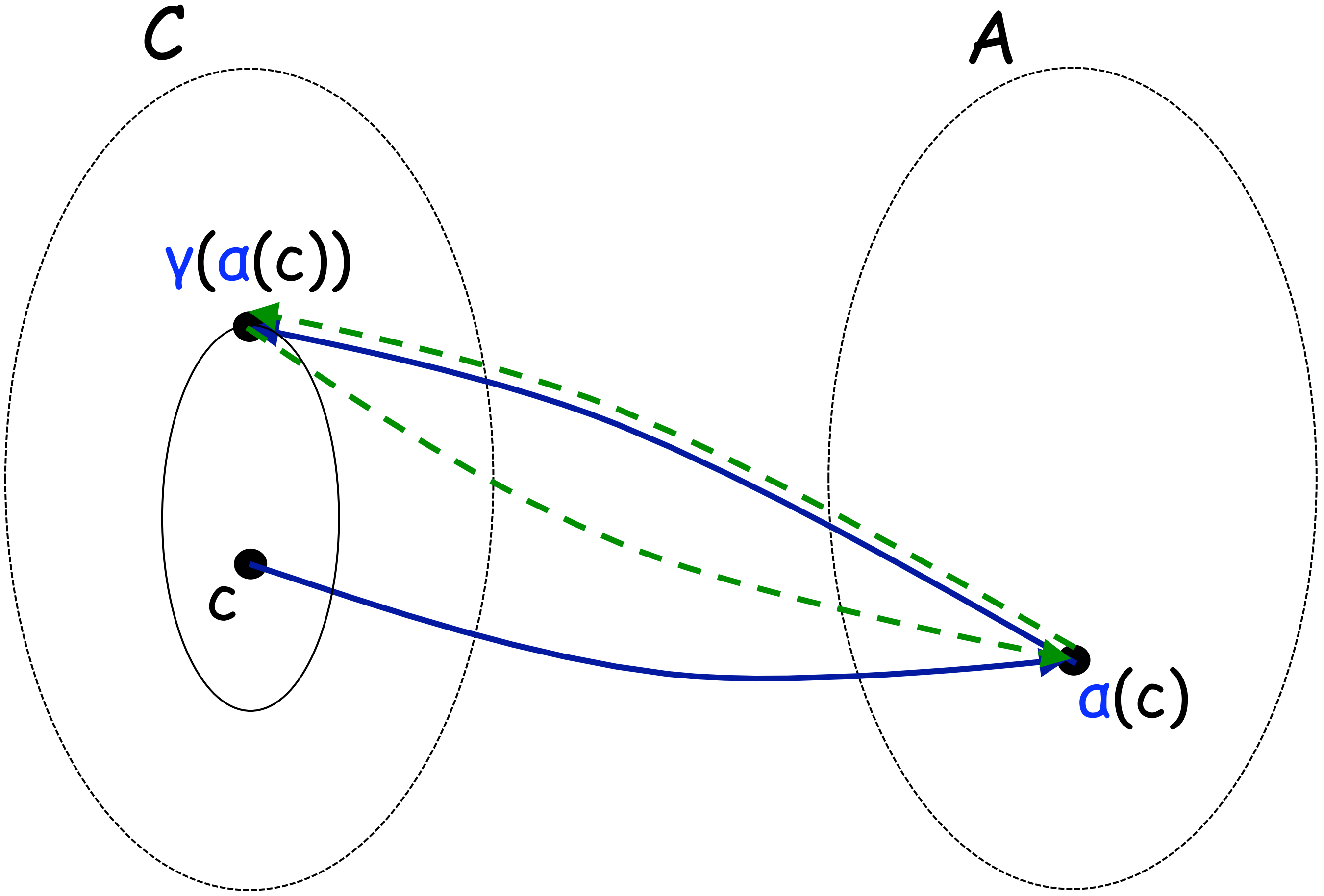
(1) α and γ are monotone functions

Galois insertion

(3) $c \subseteq \gamma(\alpha(c))$ for all $c \in \wp(C)$

(2) $a = \alpha(\gamma(a))$ for all $a \in A$

Figure: Galois insertion



Induced Operations

- A Galois connection can be used to induce the abstract operations from the concrete ones.

$$\begin{tikzcd}
A \arrow[r, "\alpha \circ op \circ \gamma"] & A \arrow[d, "\gamma"] \\
\wp(C) \arrow[u, "\alpha"] \arrow[r, "op"] & \wp(C)
\end{tikzcd}$$

abstract execution

concrete execution

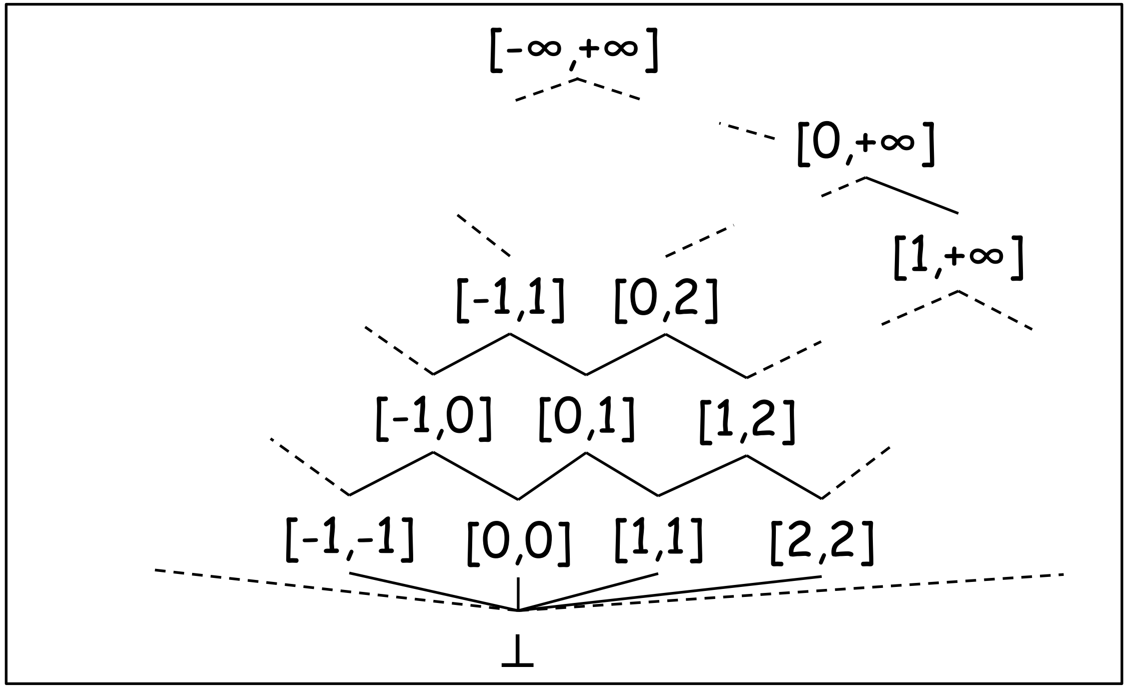
- We can show that the induced operation $\underline{op} = \alpha \circ op \circ \gamma$ is the most precise abstract operation in this setting.
- The induced operation might not be computable. In this case we can define an upper approximation $op^{\#}$, $\underline{op} \sqsubseteq op^{\#}$, and use this as abstract operation.

Abstract Interpretation

Widening

Range analysis

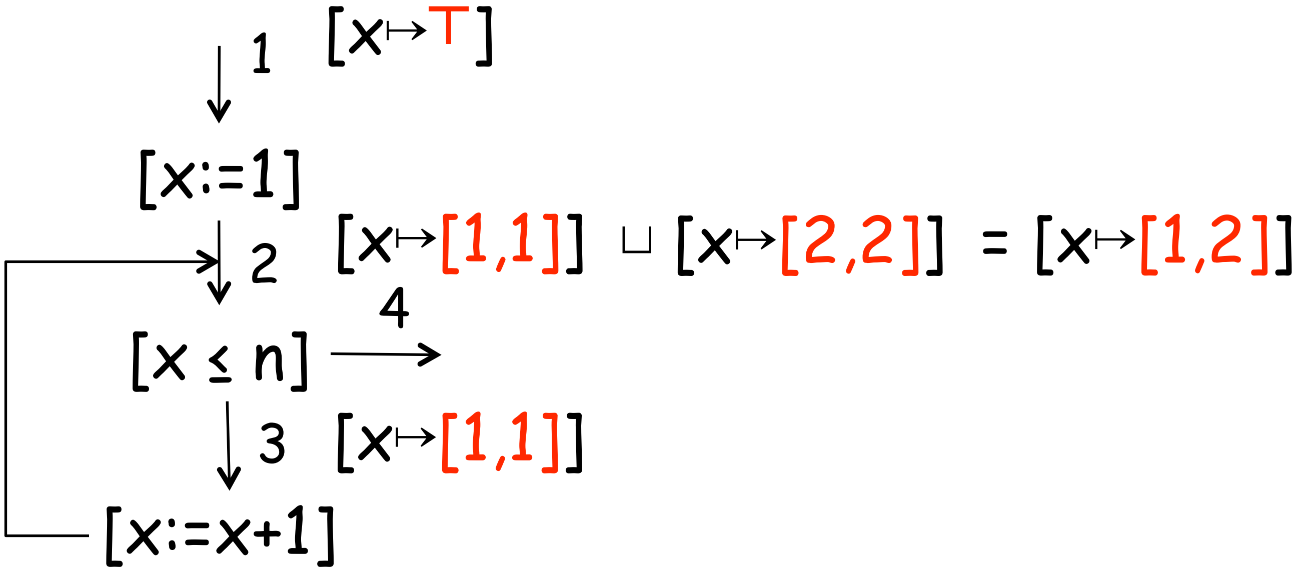
- To introduce the notion of widening, we have a look at range analysis, which provides for every variable an over-approximation of its integer value range.
- We are left with the task of choosing a suitable abstract domain: the interval lattice suggests itself.



$$\text{Interval} = \{\perp\} \cup \{[x,y] \mid x \le y,\ x \in \mathbf{Z} \cup \{\infty\},\ y \in \mathbf{Z} \cup \{\infty\}\}$$

Example

Consider the following program:



1 $[x \mapsto \top]$

[x:=1]

2 $[x \mapsto [1,1]] \sqcup [x \mapsto [2,2]] = [x \mapsto [1,2]]$

$[x \leq n]$ 4

3 $[x \mapsto [1,1]]$

[x:=x+1]

- At program point 2, the following sequence of abstract states arises: $[x \mapsto [1,1]]$, $[x \mapsto [1,2]]$, $[x \mapsto [1,3]]$, …

Consequence: The analysis never terminates (or, if n is statically known, converges only very slowly).

The ascending chain condition

- Using an arbitrary complete lattice as abstract domain, the solution is not computable in general.
- The reason for that is the fact that the value space might be unbounded, containing infinite ascending chains:

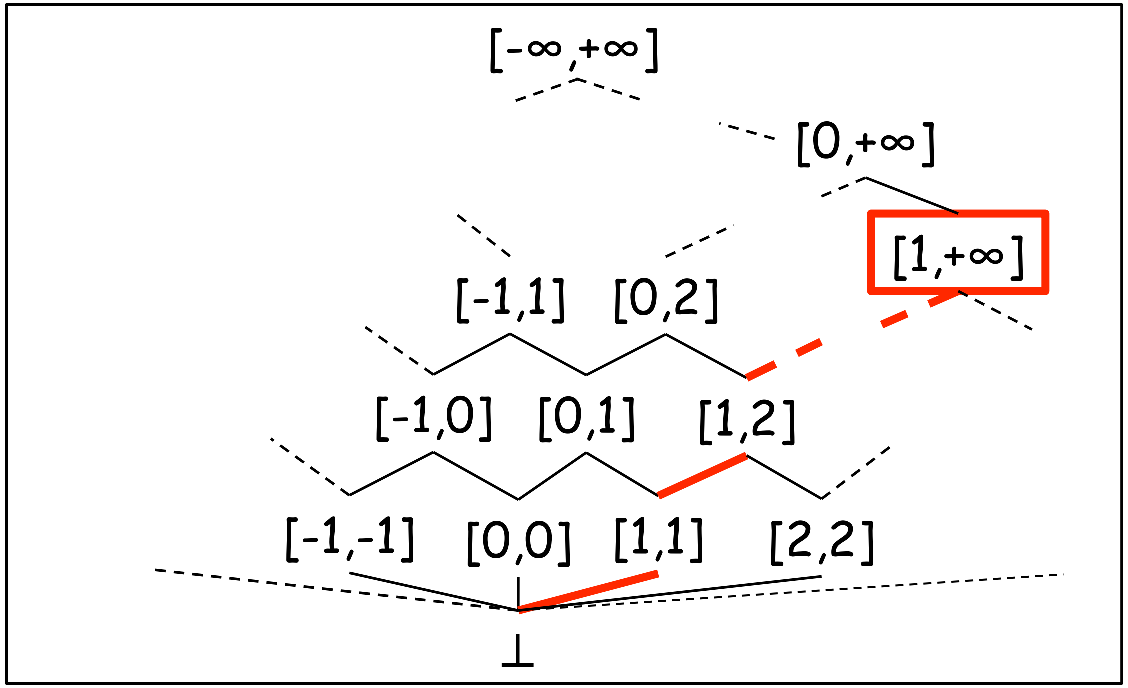
  $(l_n)_n$ is such that $l_1 \sqsubseteq l_2 \sqsubseteq l_3 \sqsubseteq \cdots$,
  but there exists *no* n such that $l_n = l_{n+1} = \cdots$

- If we replace it with an abstract space that is finite (or does not possess infinite ascending chains), then the computation is guaranteed to terminate.
- In general, we want an abstract domain to satisfy the ascending chain condition, i.e. each ascending chain eventually stabilises:

  if $(l_n)_n$ is such that $l_1 \sqsubseteq l_2 \sqsubseteq l_3 \sqsubseteq \cdots$,
  then there exists n such that $l_n = l_{n+1} = \cdots$

Non-termination

- The reason for the non-termination in the example is that the interval lattice contains infinite ascending chains.



- **Trick, if we cannot eliminate ascending chains:** We redefine the join operator of the lattice to jump to the extremal value more quickly.

Before: [1,1] ⊔ [2,2] = [1,2]   Now: [1,1] ∇ [2,2] = [1,+∞]

Widening

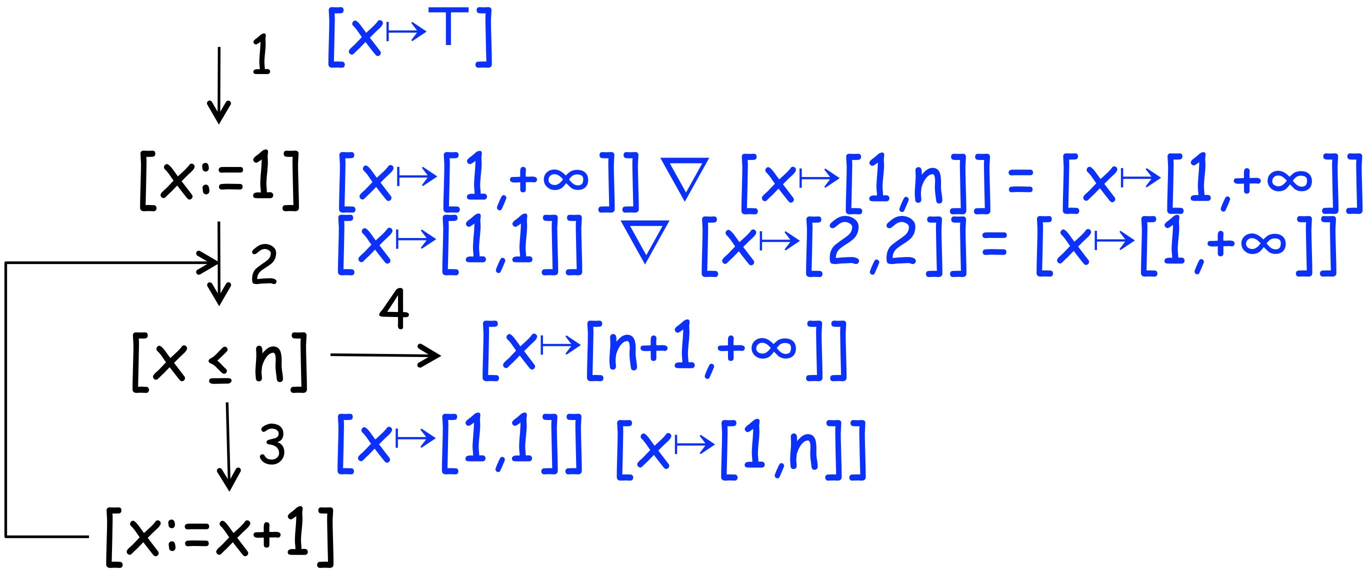
A widening $\nabla : D \times D \rightarrow D$ on a partially ordered set $(D, \sqsubseteq)$ satisfies the following properties:

1. For all $x, y \in D$. $x \sqsubseteq x \nabla y$ and $y \sqsubseteq x \nabla y$
2. For all ascending chains $x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \cdots$ the ascending chain $y_1 = x_1 \sqsubseteq y_2 = y_1 \nabla x_2 \sqsubseteq \cdots \sqsubseteq y_{n+1} = y_n \nabla x_{n+1}$ eventually stabilizes.

- Widening is used to accelerate the convergence towards an upper approximation of the least fixed point.

Example (continued)

- Assume we have a widening operator ∇ that is defined such that $[1,1] \nabla [2,2] = [1, +\infty]$



1 $[x \mapsto \top]$

[x:=1] $[x \mapsto [1,+\infty]] \nabla [x \mapsto [1,n]] = [x \mapsto [1,+\infty]]$

2 $[x \mapsto [1,1]] \nabla [x \mapsto [2,2]] = [x \mapsto [1,+\infty]]$

[x ≤ n] 4 $[x \mapsto [n+1,+\infty]]$

3 $[x \mapsto [1,1]]$ $[x \mapsto [1,n]]$

[x:=x+1]

- The analysis converges quickly.

Reading

Patrick Cousot and Radhia Cousot. *Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints*. In: POPL'77, pages 238-252. ACM Press, 1977

Neil D. Jones, Flemming Nielson: *Abstract Interpretation: a Semantics-Based Tool for Program Analysis*, 1994

Flemming Nielson, Hanne Riis Nielson, Chris Hankin: *Principles of Program Analysis*, Springer, 2005.

Chapter 1: Section 1.5

Chapter 4 (advanced material)