On the Expressiveness of MTL Variants over Dense Time

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Abstract. The basic modal operator *bounded until* of Metric Temporal Logic (MTL) comes in several variants. In particular it can be *strict* (when it does not constrain the current instant) or not, and *matching* (when it requires its two arguments to eventually hold together) or not. This paper compares the relative expressiveness of the resulting MTL variants over dense time. We prove that the expressiveness is not affected by the variations when considering non-Zeno interpretations and arbitrary nesting of temporal operators. On the contrary, the expressiveness changes for flat (i.e., without nesting) formulas, or when Zeno interpretations are allowed.

1 Introduction

In the last few decades, the formal description and analysis of real-time systems has become an increasingly important research topic. This has resulted, among other things, in the development of several formal notations for the description of real-time properties and systems. In particular, a significant number of extensions of classical temporal logics to deal with metric (quantitative) time has been introduced and used (see e.g., [3]). Among them, Metric Temporal Logic (MTL) [23, 4] is one of the most popular. An appealing feature of MTL is its being a straightforward extension of well-known Linear Temporal Logic (LTL), a classical temporal logic. In MTL, an interval parameter is added to LTL's modal operators (such as the *until* operator); the interval specifies a range of distances over which the arguments of the modality must hold, thus allowing the expression of real-time properties.

When MTL formulas are interpreted over discrete time domains (e.g., $\mathbb{N}, \mathbb{Z}$), the well-known results and techniques about the expressiveness of LTL can often be "lifted" to the real-time case [4]. On the contrary, when MTL formulas are interpreted over dense time domains (e.g., $\mathbb{R}$) additional difficulties and complications are commonly encountered, which require novel techniques (e.g., [20, 21, 5, 11, 29, 28, 6]). Another aspect where the use of MTL (and temporal logics in general) over metric dense-time models shows a substantial difference with respect to discrete time is in the *robustness* of the language expressiveness with respect to changes in its (syntactic) definitions or in the choice of the underlying

interpretation structures. In other words, it is often the case that apparently minimal changes in the definition of the basic modal operators, or in the choice of the interpretation structures (e.g., timed words rather than timed interval sequences), of the logics yield substantial differences in the resulting expressiveness. Also, these differences are usually more difficult to predict and assess than in the discrete-time case. One significant example is the use of a "natural" extension such as the introduction of past operators: it is well-known that adding them does not change the expressive power over discrete time [12] (while it increases the succinctness [25]), but it does over dense time both for LTL [18, 22], and for MTL [5, 29] (Alur and Henzinger [2] were the first to analyze this issue for a metric MTL subset known as MITL).

This paper contributes to enriching the emerging picture about the expressiveness of MTL and its common variants. The reference interpretation structure is the *behavior*, that is generic mappings that associate with every instant of time the propositions that are true at that instant. When behaviors are restricted to be non-Zeno [19] (also called *finitely variable* [30, 20]) they are an equivalent way of expressing the well-known timed interval sequences. We consider two basic language features to be varied in MTL definitions: strictness and matchingness. The basic *until* operator $\mathsf{U}(\phi_1, \phi_2)$ is called *strict* (in its first argument) if it does not constrain its first argument ϕ_1 to hold at the current instant (i.e., it constraints strictly the future); on the other hand the same operator is called *matching* if it requires the second argument ϕ_2 to hold *together with* the first argument ϕ_1 at some instant in the future (see Section 2 for precise definitions). The most common definition uses an *until* operator that is strict and non-matching; it is simple to realize that this does not restrict the expressiveness as the matching and non-strict *until*s are easily expressible in terms of strict non-matching *until*s. However, some applications dealing with MTL or closely related languages are based on the matching (e.g., [26, 27]) or non-strict (e.g., [15]) variants, or both. Therefore it is interesting to analyze if these syntactic restrictions imply restrictions in the expressiveness of the language; this is done in Section 3.

Another dimension that we consider in our analysis is the restriction to *flat* MTL formulas, i.e., formulas that do not nest temporal operators. These have also been used, among others, in some previous work of ours [15], as well as in several works with classical (qualitative) temporal logic (see related works). It is an easy guess that flat MTL is less expressive than its full "nesting" counterpart; in this paper (Section 4) we prove this intuition and then we analyze how the relationships between the various (non-)matching and (non-)strict variants change when restricted to flat formulas. In order to do so, we develop techniques to handle two different definitions in the satisfaction semantics of formulas: initial satisfiability — where the truth of a formula is evaluated only at some initial time (i.e., 0) — and global satisfiability — where the truth of a formula is evaluated at all time instants. While the two semantics are easily reconcilable when nesting is allowed, passing from the initial semantics to the global one with flat formulas is more challenging. We consider also the less common global satisfiability semantics because the expressiveness of the flat fragment is non-trivial

under such semantics (in fact, it corresponds to an implicit nesting of a qualitative temporal operator), and most common real-time properties such as bounded response and bounded invariance [23] can be easily expressed.

Finally, in Section 5, we also consider what happens to (relative) expressiveness if Zeno behaviors are allowed. In particular, we show that some equivalences between MTL variations that hold over non-Zeno behaviors are no more valid with Zeno behaviors. In this sense, allowing Zeno behaviors weakens the robustness of MTL expressiveness with respect to definitions, and it renders the picture more complicated and less intuitive.

For lack of space, we have omitted several proofs and details from this version of the paper; the interested reader can find them in [17].

Related works. As mentioned above, in recent years several works have analyzed the expressiveness of different MTL variants over dense time. Let us recall briefly the main results of a significant subset thereof.

Bouyer et al. [5] compare the expressiveness of MTL with that of TPTL (another real-time temporal logic) and they are the first to prove the conjecture that the latter is strictly more expressive than the former, over both timed words and timed interval sequences. As a corollary of their results, they show that past operators increase the expressiveness of MTL.

Since the work of Alur and Henzinger [4] it is known that (full) MTL is undecidable over dense-time models. This shortcoming has been long attributed to the possibility of expressing punctual (i.e., exact) timing constraints; in fact Alur et al. [1] have shown that MITL, a MTL subset where punctual intervals are disallowed, is decidable. However, punctuality does not always entail undecidability. In fact, Ouaknine and Worrel [28] have been the first to prove that MTL is decidable over finite timed words, albeit with non-primitive recursive complexity; their proofs rely on automata-based techniques, and in particular on the notion of timed automata with alternation. On the other hand, they show that several significant fragments of MTL are still undecidable over infinite timed words. In the same vein, Bouyer et al. [6] have identified significant MTL fragments that are instead decidable (with primitive complexity) even if one allows the expression of punctual timing constraints.

D'Souza and Prabhakar [11] compare the expressiveness of MTL over the two interpretation structures of timed words and timed interval sequences (more precisely, a specialization of the latter called "continuous" semantics). Building upon Ouaknine and Worrel's decidability results for MTL [28], they show that MTL is strictly more expressive over timed interval sequences than it is over timed words. The same authors [29] analyze a significant number of MTL variations, namely those obtained by adding past operators or by considering qualitative operators rather than metric ones, over both timed words and timed interval sequences, both in their finite and infinite forms. Still the same authors [10] have shown how to rewrite MTL formulas in flat form, and without past operators, by introducing additional propositions (a similar flattening has been shown for another temporal logic in [14]). While these latter results do not pertain directly to the expressiveness of the language (because of the new

propositions that are introduced) they help assessing the decidability of MTL variations.

In previous work [16] we proved the equivalence between the strict and non-strict non-matching variants of MTL over non-Zeno behaviors and with arbitrarily nested formulas. Section 3 uses techniques similar to those in [16] to prove new equivalence results.

These results, which we do not report with further details for the lack of space, show how relative expressiveness relations are much more complicated over dense time than they are over discrete time. In fact, some authors (e.g., [20, 3]) have suggested that these additional difficulties are an indication that the "right" semantic model for dense time has not been found yet. In particular, Hirshfeld and Rabinovich [20, 21] have made a strong point that most approaches to the definition of temporal logics for real-time and to their semantics depart from the "classical" approach to temporal logic and are too *ad hoc*, which results in unnecessary complexity and lack of robustness. While we agree with several of their remarks, we must also acknowledge that MTL (and other similar logics) has become a popular notation, and it has been used in several works. As a consequence, it is important to assess precisely the expressiveness of the language and of its common variants because of the impact on the scope of those works, even if focusing on different languages might have opened the door to more straightforward approaches.

Finally, let us mention that several works dealing with classical (qualitative) temporal logic considered variants in the definition of the basic modalities, and their impact on expressiveness and complexity. For instance, Demri and Schnoebelen [9] thoroughly investigate the complexity of LTL without nesting, or with a bounded nesting depth. Also, several works have given a very detailed characterization of how the expressiveness of LTL varies with the number of nested modalities [13, 32, 24]; and several other works, such as [8, 7], have used and characterized flat fragments where nesting is only allowed in the second argument of any *until* formula. Reynolds [31] has proved that, over dense time, LTL with strict until is strictly more expressive than LTL with a variant of non-strict until which includes the current instant. Note that Reynold's non-strict until has a different (weaker) semantics than the one we consider in this paper, because of the restriction to include the current instant. In other words, according to the notation that we introduce in Section 2, [31] compares the strict $\widetilde{\mathsf{U}}_{(0,+\infty)}$ to the non-strict $\mathsf{U}_{[0,+\infty)}$; as a consequence, Reynold's result is orthogonal to ours.

2 MTL and Its Variants

MTL is built out of the single modal operator bounded *until*[1] through propositional composition. Formulas are built according to the grammar: $\phi ::= \mathsf{p} \mid \widetilde{\mathsf{U}}_I(\phi_1, \phi_2) \mid \neg\phi \mid \phi_1 \wedge \phi_2$ where I is an interval $\langle l, u\rangle$ of the reals such that $0 \leq l \leq u \leq +\infty$, $l \in \mathbb{Q}$, $u \in \mathbb{Q} \cup \{+\infty\}$, and $\mathsf{p} \in \mathcal{P}$ is some atomic proposition from a finite set $\mathcal{P}$.

[1] In this paper we consider MTL with future operators only.

The tilde in $\widetilde{\mathsf{U}}_I$ denotes that the *until* is *strict*, as it will be apparent in the definition of its semantics; $\widetilde{\mathsf{U}}_I$ is also meant to be *non-matching*. We denote the set of formulas generated by the grammar above as $\widetilde{\mathrm{MTL}}$, which is therefore strict non-matching.

We define formally the semantics of $\widetilde{\mathrm{MTL}}$ over generic Boolean behaviors. Given a time domain $\mathbb{T}$ and a finite set of atomic propositions $\mathcal{P}$, a *Boolean behavior* over $\mathcal{P}$ is a mapping $b : \mathbb{T} \to 2^{\mathcal{P}}$ from the time domain to subsets of $\mathcal{P}$: for every time instant $t \in \mathbb{T}$, b maps t to the set of propositions $b(t)$ that are true at t. We denote the set of all mappings for a given set $\mathcal{P}$ as $\mathcal{B}_{\mathcal{P}}$, or simply as $\mathcal{B}$. In practice, in this paper we take $\mathbb{T}$ to be the reals $\mathbb{R}$, but all our results hold also for $\mathbb{R}_{\geq 0}, \mathbb{Q}, \mathbb{Q}_{\geq 0}$ as time domains.[2]

The semantics of $\widetilde{\mathrm{MTL}}$ formulas is given through a satisfaction relation $\models_{\mathbb{T}}$: given a behavior $b \in \mathcal{B}$, an instant $t \in \mathbb{T}$ (sometimes called "current instant") and an $\widetilde{\mathrm{MTL}}$ formula ϕ, the satisfaction relation is defined inductively as follows.

$b(t) \models_{\mathbb{T}} \mathsf{p}$ iff $\mathsf{p} \in b(t)$

$b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_I(\phi_1, \phi_2)$ iff there exists $d \in I$ such that $b(t+d) \models_{\mathbb{T}} \phi_2$ and, for all $u \in (0, d)$ it is $b(t+u) \models_{\mathbb{T}} \phi_1$

$b(t) \models_{\mathbb{T}} \neg\phi$ iff $b(t) \not\models_{\mathbb{T}} \phi$

$b(t) \models_{\mathbb{T}} \phi_1 \wedge \phi_2$ iff $b(t) \models_{\mathbb{T}} \phi_1$ and $b(t) \models_{\mathbb{T}} \phi_2$

From these definitions, we introduce initial satisfiability and global satisfiability as follows: a formula ϕ is *initially satisfiable* over a behavior b iff $b(0) \models_{\mathbb{T}} \phi$; a formula ϕ is *globally satisfiable* over a behavior b iff $\forall t \in \mathbb{T} : b(t) \models_{\mathbb{T}} \phi$, and we write $b \models_{\mathbb{T}} \phi$. The initial and global satisfiability relations allow one to identify a formula ϕ with the set of behaviors $[\![\phi]\!]_{\mathbb{T}}$ that satisfy it according to each semantics; hence we introduce the notation $[\![\phi]\!]^0_{\mathbb{T}} = \{b \in \mathcal{B} \mid b(0) \models_{\mathbb{T}} \phi\}$ and $[\![\phi]\!]_{\mathbb{T}} = \{b \in \mathcal{B} \mid b \models_{\mathbb{T}} \phi\}$.

From the basic strict operator we define syntactically some *variants*: the non-strict non-matching *until* U_I, the strict matching *until* $\widetilde{\mathsf{U}}^{\downarrow}_I$, and the non-strict matching *until* $\mathsf{U}^{\downarrow}_I$; they are defined in Table 1. Also, we define the following derived modal operators:[3] $\widetilde{\mathsf{R}}^{\downarrow}_I(\phi_1, \phi_2) \equiv \neg\widetilde{\mathsf{U}}^{\downarrow}_I(\neg\phi_1, \neg\phi_2)$, $\widetilde{\Diamond}_I(\phi) \equiv \widetilde{\mathsf{U}}_I(\top, \phi)$, $\widetilde{\Box}_I(\phi) \equiv \neg\widetilde{\Diamond}_I(\neg\phi)$, $\bigcirc(\phi) \equiv \mathsf{U}_{(0,+\infty)}(\phi, \top)$, and $\widetilde{\bigcirc}(\phi) \equiv \widetilde{\mathsf{U}}_{(0,+\infty)}(\phi, \top)$; derived propositional connectives (such as $\Rightarrow, \vee, \Leftrightarrow$) are defined as usual. For derived operators we use the same notational conventions: a $\sim$ denotes strictness and a $\downarrow$ denotes matchingness. Accordingly, we denote by MTL the set of non-strict non-matching formulas (i.e., those using only the U_I operator), by $\widetilde{\mathrm{MTL}}^{\downarrow}$ the set of strict matching formulas (i.e., those using only the $\widetilde{\mathsf{U}}^{\downarrow}_I$ operator), and by $\mathrm{MTL}^{\downarrow}$ the set of non-strict matching formulas (i.e., those using only the $\mathsf{U}^{\downarrow}_I$ operator).

Note that the $\widetilde{\Diamond}$ operator (and correspondingly the $\widetilde{\Box}$ operator as well) can be equivalently expressed with any of the *until* variants introduced beforehand,

[2] Even if we deal only with future operators, bi-infinite time domains $\mathbb{R}$ and $\mathbb{Q}$ are considered as they match "more naturally" the global satisfiability semantics.

[3] For clarity, let us give explicitly the semantics of the $\widetilde{\mathsf{R}}^{\downarrow}_I$ operator: $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{R}}^{\downarrow}_I(\phi_1, \phi_2)$ iff for all $d \in I$ it is: $b(t+d) \models_{\mathbb{T}} \phi_2$ or $b(t+u) \models_{\mathbb{R}} \phi_1$ for some $u \in (0, d]$.

i.e., $\widetilde{\Diamond}_I(\phi) \equiv \widetilde{\mathsf{U}}_I(\top,\phi) \equiv \mathsf{U}_I(\top,\phi) \equiv \widetilde{\mathsf{U}}^{\downarrow}_I(\top,\phi) \equiv \mathsf{U}^{\downarrow}_I(\top,\phi)$. Therefore, in the following we drop the tilde and write $\Diamond_I$ (resp. $\Box_I$) in place of $\widetilde{\Diamond}_I$ (resp. $\widetilde{\Box}_I$).

Operator	$\equiv$	Definition
$\mathsf{U}_I(\phi_1,\phi_2)$	$\equiv$	if $0 \notin I$: $\phi_1 \wedge \widetilde{\mathsf{U}}_I(\phi_1,\phi_2)$ else: $\phi_2 \vee (\phi_1 \wedge \widetilde{\mathsf{U}}_I(\phi_1,\phi_2))$
$\widetilde{\mathsf{U}}^{\downarrow}_I(\phi_1,\phi_2)$	$\equiv$	if $0 \notin I$: $\widetilde{\mathsf{U}}_I(\phi_1,\phi_2 \wedge \phi_1)$ else: $\phi_2 \vee (\widetilde{\mathsf{U}}_I(\phi_1,\phi_2 \wedge \phi_1))$
$\mathsf{U}^{\downarrow}_I(\phi_1,\phi_2)$	$\equiv$	$\phi_1 \wedge \widetilde{\mathsf{U}}_I(\phi_1,\phi_2 \wedge \phi_1) \equiv \mathsf{U}_I(\phi_1,\phi_2 \wedge \phi_1)$

Table 1. *Until* operator variants.

According to the semantics, all formulas of some MTL variant identify a set of sets of behaviors which characterize the *expressive power* of that variant. We overload the notation and also denote by $\widetilde{\text{MTL}}$, MTL, $\widetilde{\text{MTL}}^{\downarrow}$, and $\text{MTL}^{\downarrow}$ the set of sets of behaviors identified by all strict non-matching, non-strict matching, strict matching, and non-strict non-matching formulas, respectively. It will be clear from the context whether we are referring to a set of formulas or to the corresponding set of sets of behaviors, and whether we are considering the initial or global satisfiability semantics.

For every formula ϕ, we define its *granularity* ρ as the reciprocal of the product of all denominators of non-null finite interval bounds appearing in ϕ; and its *nesting depth* (also called *temporal height*) k as the maximum number of nested modalities in ϕ. A formula is called *flat* if it does not nest modal operators, and *nesting* otherwise. Given a set of formulas F, the subset of all its flat formulas is denoted by $\flat F$ (for instance flat non-strict non-matching formulas are denoted as $\flat$MTL).

Since the non-strict and matching variants have been defined in terms of $\widetilde{\text{MTL}}$ — and their definitions do not nest temporal operators — it is clear that the following relations hold: $\text{MTL}^{\downarrow} \subseteq \text{MTL} \subseteq \widetilde{\text{MTL}}$, $\text{MTL}^{\downarrow} \subseteq \widetilde{\text{MTL}}^{\downarrow} \subseteq \widetilde{\text{MTL}}$, $\flat\text{MTL}^{\downarrow} \subseteq \flat\text{MTL} \subseteq \flat\widetilde{\text{MTL}}$, and $\flat\text{MTL}^{\downarrow} \subseteq \flat\widetilde{\text{MTL}}^{\downarrow} \subseteq \flat\widetilde{\text{MTL}}$.

Non-Zenoness. Behaviors over dense time are often subject to the *non-Zenoness* (also called *finite variability* [20, 30]) requirement [19]. A behavior $b \in \mathcal{B}$ is called *non-Zeno* if the truth value of any atomic proposition $\mathsf{p} \in \mathcal{P}$ changes in b only finitely many times over any bounded interval of time. In [16] we proved that strict $\widetilde{\bigcirc}$ operator can be expressed with non-strict $\bigcirc$ operator over non-Zeno behaviors as $\widetilde{\bigcirc}(\phi) \equiv \bigcirc(\phi) \vee (\neg\phi \wedge \neg \bigcirc(\neg\phi))$.

3 Nesting MTL over non-Zeno Behaviors

This section shows that the four MTL variants: $\widetilde{\text{MTL}}$, MTL, $\widetilde{\text{MTL}}^{\downarrow}$, and $\text{MTL}^{\downarrow}$ all have the same expressive power over non-Zeno behaviors, for both the initial and global satisfiability semantics. In fact, we provide a set of equivalences

according to which one can replace each occurrence of strict *until* in terms of non-strict *until*, and each occurrence of non-matching *until* in terms of matching *until*. This shows that $\mathrm{MTL} = \widetilde{\mathrm{MTL}} = \widetilde{\mathrm{MTL}}^{\downarrow} = \mathrm{MTL}^{\downarrow}$. Note that the result holds regardless of whether the global or initial satisfiability relation in considered.

3.1 Non-Strict as Expressive as Strict

In [16] we have shown that $\mathrm{MTL} = \widetilde{\mathrm{MTL}}$; more precisely, the following equivalences have been proved, for $a > 0$ (and $b > 0$ in (4)).

$$\widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2) \;\equiv\; \Diamond_{(a,b\rangle}(\phi_2) \wedge \Box_{(0,a]}\Big(\mathsf{U}_{(0,+\infty)}(\phi_1, \phi_2)\Big) \tag{1}$$

$$\widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2) \;\equiv\; \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2) \vee \Big(\Box_{(0,a)}(\phi_1) \wedge \Diamond_{=a}(\phi_2)\Big) \tag{2}$$

$$\widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2) \;\equiv\; \Diamond_{(0,b\rangle}(\phi_2) \wedge \widetilde{\bigcirc}\Big(\mathsf{U}_{(0,+\infty)}(\phi_1, \phi_2)\Big) \tag{3}$$

$$\widetilde{\mathsf{U}}_{[0,b\rangle}(\phi_1, \phi_2) \;\equiv\; \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2) \vee \phi_2 \tag{4}$$

$$\widetilde{\mathsf{U}}_{[0,0]}(\phi_1, \phi_2) \;\equiv\; \phi_2 \tag{5}$$

(1–5) provide a means to replace each occurrence of strict *until* with non-strict *until*s only. Also, if we replace each occurrence of formula ϕ_2 in (1–5) with $\phi_2 \wedge \phi_1$ — except for (4) which requires a slightly different treatment, which is however routine — we also have a proof that $\widetilde{\mathrm{MTL}}^{\downarrow} = \mathrm{MTL}^{\downarrow}$, according to the definition of the matching variants of the *until* operators.

3.2 Matching as Expressive as Non-Matching

This section provides a set of equivalences to replace each occurrence of a strict non-matching operator with a formula that contains only strict matching operators; this shows that $\widetilde{\mathrm{MTL}} = \widetilde{\mathrm{MTL}}^{\downarrow}$. To this end, let us first prove the following equivalence.

$$\begin{aligned} \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2) \quad \equiv \quad & \widetilde{\mathsf{U}}^{\downarrow}_{(0,b\rangle}(\phi_1, \phi_2) \\ & \vee\; (\Diamond_{(0,b\rangle}(\phi_2) \wedge \widetilde{\bigcirc}(\phi_1) \wedge \widetilde{\mathsf{R}}^{\downarrow}_{(0,b\rangle}(\phi_2, \bigcirc(\phi_1)) \end{aligned} \tag{6}$$

Proof (of Formula 6). Let us start with the $\Leftarrow$ direction, and let t be the current instant. If $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{U}}^{\downarrow}_{(0,b\rangle}(\phi_1, \phi_2)$ clearly also $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2)$ *a fortiori*. So let us assume that $\widetilde{\mathsf{U}}^{\downarrow}_{(0,b\rangle}(\phi_1, \phi_2)$ is false at t; note that this subsumes that $b(t) \models_{\mathbb{R}} \neg\widetilde{\bigcirc}(\phi_2 \wedge \phi_1)$.

Let us remark that we can assume that $\widetilde{\bigcirc}(\neg\phi_2)$ holds at t, because $\widetilde{\bigcirc}(\phi_1)$ and $\neg\widetilde{\bigcirc}(\phi_2 \wedge \phi_1)$ both hold. Therefore, it is well-defined u, the smallest instant in $(t, t+b\rangle$ such that $b(u) \models_{\mathbb{R}} \phi_2 \vee \widetilde{\bigcirc}(\phi_2)$. Note that this implies that ϕ_2 is false throughout $(t, u\rangle$, with the interval right-open iff ϕ_2 holds at u.

Let us first consider the case $b(u) \models_{\mathbb{R}} \phi_2$. Let v be a generic instant in (t, u); recall that ϕ_2 is false throughout $(t, u) \supset (t, v]$. Therefore it must be $b(v) \models_{\mathbb{R}} \bigcirc(\phi_1)$ for $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{R}}^{\downarrow}_{(0,b\rangle}(\phi_2, \bigcirc(\phi_1))$ to be true. So, ϕ_1 holds throughout (t, u) and ϕ_2 holds at u, which means that $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2)$.

Let us now consider the other case $b(u) \models_{\mathbb{R}} \neg\phi_2 \wedge \widetilde{\bigcirc}(\phi_2)$. Let v be a generic instant in $(t, u]$; recall that ϕ_2 is false throughout $(t, u] \supseteq (t, v]$. From $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{R}}^{\downarrow}_{(0,b\rangle}(\phi_2, \bigcirc(\phi_1))$ it must be $b(v) \models_{\mathbb{R}} \bigcirc(\phi_1)$. Overall, ϕ_1 holds throughout $(t, u+\epsilon]$ for some $\epsilon > 0$, as in particular $b(t+u) \models_{\mathbb{R}} \bigcirc(\phi_1)$. Clearly, this subsumes $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2)$.

For brevity, we omit the simpler $\Rightarrow$ direction (see [17] for details). □

The case for $a > 0$ can be handled simply by relying on the previous equivalence. In fact, the following equivalence is easily seen to hold.

$$\begin{aligned} \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2) \quad &\equiv \quad \widetilde{\mathsf{U}}^{\downarrow}_{(a,b\rangle}(\phi_1, \phi_2) \\ &\quad \vee \ \Big(\Box_{(0,a]}(\phi_1) \wedge \Diamond_{=a}\Big(\widetilde{\mathsf{U}}_{(0,b-a\rangle}(\phi_1, \phi_2)\Big)\Big) \end{aligned} \tag{7}$$

The cases for left-closed intervals are also derivable straightforwardly as:

$$\widetilde{\mathsf{U}}_{[0,b\rangle}(\phi_1, \phi_2) \quad \equiv \quad \phi_2 \vee \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2) \tag{8}$$

and

$$\widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2) \quad \equiv \quad \Big(\Diamond_{=a}(\phi_2) \wedge \Box_{(0,a)}(\phi_1)\Big) \vee \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2) \tag{9}$$

Finally, let us note that the $\bigcirc(\phi)$ operator can be expressed equivalently with strict matching operators as $\phi \wedge \widetilde{\mathsf{U}}^{\downarrow}_{(0,+\infty)}(\phi, \top)$. In fact, $\bigcirc(\phi)$ at x means that ϕ holds over an interval $[x, x+\epsilon)$ for some $\epsilon > 0$; therefore, ϕ also holds over a closed interval such as $[x, x+\epsilon/2]$, as required by $\phi \wedge \widetilde{\mathsf{U}}^{\downarrow}_{(0,+\infty)}(\phi, \top)$, and *vice versa*.

All in all (6–9) provide a means to replace every occurrence of strict non-matching *until* with a formula that contains only strict matching *until*s. This shows that $\widetilde{\mathrm{MTL}} = \widetilde{\mathrm{MTL}}^{\downarrow}$, completing our set of equivalences for non-Zeno behaviors.

4 Flat MTL

Section 3 has shown the equivalence of all (non-)strict and (non-)matching MTL variants for non-Zeno behaviors. It is apparent, however, that the equivalences

between the various *until* variants introduce nesting of temporal operators, that is they change flat formulas into nesting ones. This section shows that this is inevitable, as the relative expressiveness relations change if we consider flat formulas only. More precisely, we prove that both non-strictness and matchingness lessen the expressive power of MTL flat formulas, so that the strict non-matching variant is shown to be the most expressive. We also show that, as one would expect, even this most expressive flat variant is less expressive than any nesting variant. All separation results are proved under both the initial satisfiability and the global satisfiability semantics.

4.1 Non-Strict Less Expressive Than Strict

This section shows that $\flat\mathrm{MTL} \subset \flat\widetilde{\mathrm{MTL}}$; let us outline the technique used to prove this fact. We provide a strict flat formula $\alpha \in \flat\widetilde{\mathrm{MTL}}$ and we prove that it has no equivalent non-strict flat formula. The proof goes adversarially: assume $\beta \in \flat\mathrm{MTL}$ is a non-strict flat formula equivalent to α, and let ρ be the granularity of β. From ρ we build two behaviors $b^\rho_\top$ and $b^\rho_\bot$ such that any $\flat$MTL formula of granularity ρ (and β in particular) cannot distinguish between them, i.e., it is either satisfied by both or by none. On the contrary, α is satisfied by $b^\rho_\top$ but not by $b^\rho_\bot$, for all ρ. This shows that no equivalent non-strict flat formula can exist, and thus $\flat\widetilde{\mathrm{MTL}} \not\subseteq \flat\mathrm{MTL}$. From $\flat\mathrm{MTL} \subseteq \flat\widetilde{\mathrm{MTL}}$ we conclude that $\flat\mathrm{MTL} \subset \flat\widetilde{\mathrm{MTL}}$.

As in all separation results, the details of the proofs are rather involved; this is even more the case when considering the global satisfiability semantics; throughout we will try to provide some intuition referring to [17] for all the lower-level details.

Let us define the following families of behaviors over $\{\mathsf{p}\}$. For any given $\rho > 0$, let $b^\rho_\top$ and $b^\rho_\bot$ be defined as follows: $\mathsf{p} \in b^\rho_\top(t)$ iff $t \leq 0$ or $t \geq \rho/4$; and $\mathsf{p} \in b^\rho_\bot(t)$ iff $t \leq 0$ or $t > \rho/4$. Similarly, for any given $\rho > 0$, $c^\rho_\bot$ is defined as follows: $\mathsf{p} \in c^\rho_\bot(t)$ iff $\mathsf{p} \in b^\rho_\bot(t)$ and $t \neq 0$. Note that $b^\rho_\top(t) = b^\rho_\bot(t)$ for all $t \neq \rho/4$, and that $b^\rho_\bot(t) = c^\rho_\bot(t)$ for all $t \neq 0$. Let us also define the sets of behaviors $\mathcal{B}_\top = \bigcup_{\rho \in \mathbb{Q}_{>0}} b^\rho_\top$ and $\mathcal{B}_\bot = \bigcup_{\rho \in \mathbb{R}_{>0}} b^\rho_\bot$. The behaviors $b^\rho_\top, b^\rho_\bot, c^\rho_\bot$ are pictured in Figure 1.

Initial satisfiability. Let us first assume the initial satisfiability semantics; we show that no $\flat$MTL formula ϕ with granularity ρ distinguishes initially between $b^\rho_\top$ and $b^\rho_\bot$. To this end, we prove the following.

Lemma 1. *For any* $\flat$MTL *formula* ϕ *of granularity* ρ*, it is* $b^\rho_\top(0) \models_{\mathbb{R}} \phi$ *iff* $b^\rho_\bot(0) \models_{\mathbb{R}} \phi$.

Proof. The proof is by induction on the structure of ϕ. Let us consider just a few most relevant cases (the others are in [17]): assume $\phi = \mathsf{U}_I(\beta_1, \beta_2)$, with $I = \langle l, u \rangle$ and: (1) $l = k_1\rho$ for some $k_1 \in \mathbb{N}$; and (2) $u = k_2\rho$ or $u = +\infty$, for some $k_1 \leq k_2 \in \mathbb{N}$. Note that we can assume $0 \notin I$ without loss of generality, as $\mathsf{U}_{[0,u\rangle}(\beta_1, \beta_2) \equiv \beta_2 \vee \mathsf{U}_{(0,u\rangle}(\beta_1, \beta_2)$. We also assume that I is non-empty; this is also without loss of generality. We then consider all cases for β_1, β_2; in particular:

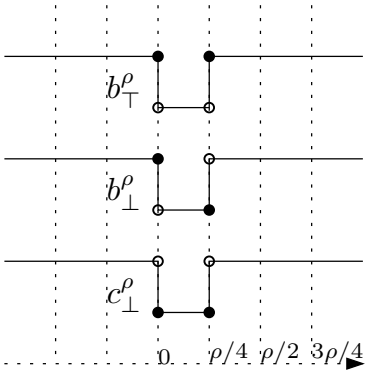

Fig. 1. The behaviors $b^\rho_\top, b^\rho_\bot, c^\rho_\bot$.

- If $\beta_1 \equiv \top$ and β_2 is one of p or $\neg\mathsf{p}$, we have $\phi \equiv \Diamond_I(\beta_2)$.
  If $l = 0$, then $u > l$, which entails $u \geq \rho$. Any interval of the form $\langle 0, \rho\rangle$ encompasses both instants where p holds and instants where $\neg\mathsf{p}$ holds. Thus, $b^\rho_\top(0) \models_{\mathbb{R}} \phi$ and $b^\rho_\bot(0) \models_{\mathbb{R}} \phi$ in this case.
  If $l > 0$ then $l \geq \rho$. Then, whatever $u \geq l$ is, it is clear that p holds throughout the non-empty interval $\langle l, u\rangle$. Therefore, if $\beta_2 \equiv \mathsf{p}$ we have $b^\rho_\top(0) \models_{\mathbb{R}} \phi$ and $b^\rho_\bot(0) \models_{\mathbb{R}} \phi$; otherwise $\beta_2 \equiv \neg\mathsf{p}$, and $b^\rho_\top(0) \not\models_{\mathbb{R}} \phi$ and $b^\rho_\bot(0) \not\models_{\mathbb{R}} \phi$.
- If $\beta_1 \equiv \mathsf{p}$ or $\beta_1 \equiv \neg\mathsf{p}$, then ϕ does not hold unless $\bigcirc(\beta_1)$ holds (still because $0 \notin I$). Since the value of p changes from 0 to its immediate future, it is $b^\rho_\top(0) \not\models_{\mathbb{R}} \phi$ and $b^\rho_\bot(0) \not\models_{\mathbb{R}} \phi$. □

Lemma 1 leads straightforwardly to the desired separation result.

Theorem 1. *Under the initial satisfiability semantics,* $\flat\mathrm{MTL} \subset \flat\widetilde{\mathrm{MTL}}$.

Proof. Let us show that the $\flat\widetilde{\mathrm{MTL}}$ formula $\Sigma = \widetilde{\mathsf{U}}_{(0,+\infty)}(\neg\mathsf{p}, \mathsf{p})$ has no equivalent $\flat$MTL formula. Σ can distinguish initially between the families of behaviors $\mathcal{B}_\top, \mathcal{B}_\bot$, as for any $b \in \mathcal{B}_\top, b' \in \mathcal{B}_\bot$, it is $b(0) \models_{\mathbb{R}} \Sigma$ and $b'(0) \not\models_{\mathbb{R}} \Sigma$. Let us assume that σ is an $\flat$MTL formula of granularity ρ equivalent to Σ. However, from Lemma 1 it follows that $b^\rho_\top(0) \models_{\mathbb{R}} \sigma$ iff $b^\rho_\bot(0) \models_{\mathbb{R}} \sigma$. Therefore, σ is not equivalent to Σ. □

Global satisfiability. Let us now prove an analogous of Theorem 1 for the global satisfiability semantics.

Lemma 2. *For any* $\flat$MTL *formula* ϕ *of granularity* ρ *and any instant* $t < 0$*, it is:* $b^\rho_\top(t) \models_{\mathbb{R}} \phi$ *iff* $b^\rho_\bot(t) \models_{\mathbb{R}} \phi$*, or* $b^\rho_\bot(t) \models_{\mathbb{R}} \phi$ *iff* $c^\rho_\bot(t) \models_{\mathbb{R}} \phi$.

Proof. The proof is by induction on the structure of ϕ; throughout, t is any fixed instant less than 0. Let us just outline a few significant cases; all details are in [17].

For the base case, if $\phi = \mathsf{U}_I(\beta_1, \beta_2)$ with $\beta_1, \beta_2 \in \{\mathsf{p}, \neg\mathsf{p}, \top, \bot\}$ one can verify that it is $b^\rho_\top \models_\mathbb{R} \phi$ iff $b^\rho_\bot \models_\mathbb{R} \phi$ iff $c^\rho_\bot \models_\mathbb{R} \phi$, unless: (a) $b^\rho_\top \not\models_\mathbb{R} \phi$ iff $b^\rho_\bot \models_\mathbb{R} \phi$ iff $c^\rho_\bot \models_\mathbb{R} \phi$ and $t + k\rho = \rho/4$ or $t + k\rho = 0$ for some positive integer k; or (b) $b^\rho_\top \models_\mathbb{R} \phi$ iff $b^\rho_\bot \models_\mathbb{R} \phi$ iff $c^\rho_\bot \not\models_\mathbb{R} \phi$ and $t + h\rho = 0$ for some positive integer h.

Therefore, consider the inductive case $\phi = \phi_1 \wedge \phi_2$; in particular let us focus on the "crucial" case $b^\rho_\top(t) \not\models_\mathbb{R} \phi_i$ iff $b^\rho_\bot(t) \models_\mathbb{R} \phi_i$ iff $c^\rho_\bot(t) \models_\mathbb{R} \phi_i$ and $t + k\rho = \rho/4$ for some i, and $b^\rho_\top(t) \models_\mathbb{R} \phi_j$ iff $b^\rho_\bot(t) \models_\mathbb{R} \phi_j$ iff $c^\rho_\bot(t) \not\models_\mathbb{R} \phi_j$ and $t + h\rho = 0$ for $j \neq i$. This case, however, is not possible as $t + k\rho = \rho/4 = \rho/4 + t + h\rho$ implies $(k - h)\rho = \rho/4$ which is impossible as k and h are integers. To give some intuition, this is due to the granularity: in other words, from the same t we cannot reference both 0 and $\rho/4$, since they are less than ρ time instants apart. Finally also note that this restriction can be "lifted" to the conjunction itself, to go with the inductive hypothesis. □

Through Lemma 2 we can extend Theorem 1 to the global satisfiability semantics.

Theorem 2. *Under the global satisfiability semantics,* $\flat\mathrm{MTL} \subset \flat\widetilde{\mathrm{MTL}}$.

Proof. Let us show that the $\flat\widetilde{\mathrm{MTL}}$ formula $\Omega = \widetilde{\mathsf{U}}_{(0,+\infty)}(\neg\mathsf{p}, \mathsf{p}) \vee \bigcirc(\neg\mathsf{p}) \vee \widetilde{\bigcirc}(\mathsf{p})$ has no equivalent $\flat$MTL formula. It is simple to check that, for all $b \in \mathcal{B}_\top, b' \in \mathcal{B}_\bot$ it is $b \models_\mathbb{R} \Omega$ and $b' \not\models_\mathbb{R} \Omega$; more precisely, it is $b'(0) \not\models_\mathbb{R} \Omega$ and, for all $t > 0$, $b'(t) \models_\mathbb{R} \Omega$. Also, for all $b'' \in \bigcup_\rho c^\rho_\bot$, it is $b'' \models_\mathbb{R} \Omega$.

Now the proof goes by *reductio ad absurdum*. Let ω be an $\flat$MTL formula of granularity ρ equivalent to Ω. Thus it must be $b^\rho_\top \models_\mathbb{R} \omega$, $b^\rho_\bot \not\models_\mathbb{R} \omega$, and $c^\rho_\bot \models_\mathbb{R} \omega$. So, there exists a t such that $b^\rho_\bot(t) \not\models_\mathbb{R} \omega$. Let us show that no such t can exist. Lemma 1 mandates that $b^\rho_\bot(0) \models_\mathbb{R} \omega$, so it must be $t \neq 0$.

If $t > 0$, recall that $c^\rho_\bot \models_\mathbb{R} \omega$. This subsumes that $c^\rho_\bot(u) \models_\mathbb{R} \omega$ for all $u > 0$, and thus in particular at t. However, note that $c^\rho_\bot(x) = b^\rho_\bot(x)$ for all $x > 0$; since ω is a future formula, its truth value to the future of 0 cannot change when just one *past* instant has changed and the future has not changed. So it must be $b^\rho_\bot(t) \models_\mathbb{R} \omega$: a contradiction.

Let us now assume $t < 0$. From Lemma 2 for formula ω, it is either (1) $b^\rho_\top(t) \models_\mathbb{R} \omega$ iff $b^\rho_\bot(t) \models_\mathbb{R} \omega$; or (2) $b^\rho_\bot(t) \models_\mathbb{R} \omega$ iff $c^\rho_\bot(t) \models_\mathbb{R} \omega$. However, $b^\rho_\top(t) \models_\mathbb{R} \omega$ and $b^\rho_\bot(t) \not\models_\mathbb{R} \omega$, so (1) is false and (2) must be true. Hence, it must be $c^\rho_\bot(t) \not\models_\mathbb{R} \omega$. But this implies $c^\rho_\bot \not\models_\mathbb{R} \omega$, whereas it should be $c^\rho_\bot \models_\mathbb{R} \omega$ since ω is supposed equivalent to Ω: a contradiction again. □

4.2 Matching Less Expressive Than Non-Matching

This section provides an indirect simple proof that $\flat\widetilde{\mathrm{MTL}}^\downarrow \subset \flat\widetilde{\mathrm{MTL}}$. To this end we first show the equivalence of non-strict and strict flat matching MTL when restricted to a unary set of propositions.

Lemma 3. *Over a unary set of propositions* $\mathcal{P} : |\mathcal{P}| = 1$, $\flat\mathrm{MTL}^\downarrow = \flat\widetilde{\mathrm{MTL}}^\downarrow$.

Proof (sketch). We can show that any $\flat\widetilde{\mathrm{MTL}}^{\downarrow}$ formula $\phi = \widetilde{\mathsf{U}}_I^{\downarrow}(\beta_1, \beta_2)$ has an equivalent $\flat\mathrm{MTL}^{\downarrow}$ formula for unary alphabet, as when $\beta_1 = \neg\beta_2$ ϕ is trivially false, according to the semantics of Section 2. □

As a corollary of Lemma 3 we can separate $\flat\widetilde{\mathrm{MTL}}^{\downarrow}$ and $\flat\widetilde{\mathrm{MTL}}$ (over general set of propositions).

Theorem 3. $\flat\widetilde{\mathrm{MTL}}^{\downarrow} \subset \flat\widetilde{\mathrm{MTL}}$.

Proof. Recall that $\flat\mathrm{MTL}^{\downarrow} \subseteq \flat\mathrm{MTL}$. Let us first assume a unary alphabet; from Lemma 3 it is $\flat\widetilde{\mathrm{MTL}}^{\downarrow} = \flat\mathrm{MTL}^{\downarrow}$. Since Theorems 1 and 2 are based on counterexamples over unary alphabet, it is also $\flat\mathrm{MTL} \subset \flat\widetilde{\mathrm{MTL}}$. All in all: $\flat\widetilde{\mathrm{MTL}}^{\downarrow} = \flat\mathrm{MTL}^{\downarrow} \subseteq \flat\mathrm{MTL} \subset \flat\widetilde{\mathrm{MTL}}$, hence $\flat\widetilde{\mathrm{MTL}}^{\downarrow} \subset \flat\widetilde{\mathrm{MTL}}$ over unary alphabet, which implies the same holds over generic alphabet. □

4.3 Non-Strict Matching Less Expressive Than Matching

Section 4.1 shows that strict flat MTL is strictly more expressive than its non-strict flat counterpart, when both of them are in their non-matching version. If we consider the matching versions of strict and non-strict operators, one can prove that the same holds, that is $\flat\mathrm{MTL}^{\downarrow} \subset \flat\widetilde{\mathrm{MTL}}^{\downarrow}$.

Lemma 3 entails that any separation proofs for $\flat\mathrm{MTL}^{\downarrow}$ and $\flat\widetilde{\mathrm{MTL}}^{\downarrow}$ must consider behaviors over alphabets of size at least two. In fact, it is possible to use a technique similar to that of Section 4.1, but with behaviors over alphabet of size two. For details we refer to [17].

Theorem 4. *Under the initial and global satisfiability semantics,* $\flat\mathrm{MTL}^{\downarrow} \subset \flat\widetilde{\mathrm{MTL}}^{\downarrow}$.

4.4 Non-Strict Matching Less Expressive Than Non-Strict

Section 4.2 shows that flat non-matching MTL is strictly more expressive than its matching flat counterpart, when both of them are in their strict version. The same relation holds if we consider the non-strict versions of matching and non-matching operators, that is we can prove that $\flat\mathrm{MTL}^{\downarrow} \subset \flat\mathrm{MTL}$. The proof technique is again is similar to the one in the previous Section 4.3, and it is based on behaviors over a binary set of propositions; see [17] for details.

Theorem 5. *Under the initial and global satisfiability semantics,* $\flat\mathrm{MTL}^{\downarrow} \subset \flat\mathrm{MTL}$.

4.5 Flat Less Expressive Than Nesting

Through a technique similar to that used in Section 4.1 it is also possible to show that $\flat\widetilde{\mathrm{MTL}} \subset \mathrm{MTL}$. For the lack of space we refer to [17] for all details.

Theorem 6. *Under the initial and global satisfiability semantics,* $\flat\widetilde{\mathrm{MTL}} \subset \mathrm{MTL}$.

5 Nesting MTL over Zeno Behaviors

This section re-considers some of the expressiveness results for nesting formulas of Section 3 when Zeno behaviors are allowed as interpretation structures, and in particular it shows that the equivalence between MTL and $\widetilde{\text{MTL}}$ does not hold if we allow Zeno behaviors.

5.1 Non-Strict Less Expressive Than Strict

Let us first show that MTL $\subset \widetilde{\text{MTL}}$ over generic behaviors. To this end, we define behaviors $b_\delta, b_\delta^{\mathcal{Z}}$ over $\mathcal{P} = \{\mathsf{p}\}$, for all $\delta > 0$. b_δ is defined as: $\mathsf{p} \in b_\delta(t)$ iff $t = k\delta/2$ for some $k \in \mathbb{Z}$. $b_\delta^{\mathcal{Z}}$ is defined as: $\mathsf{p} \in b_\delta^{\mathcal{Z}}(t)$ iff $t = (k + 2^{-n})\delta$, for some $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Clearly, for all $t \in \mathbb{T}$, $\mathsf{p} \in b_\delta(t)$ implies $\mathsf{p} \in b_\delta^{\mathcal{Z}}(t)$; moreover, notice that $b_\delta^{\mathcal{Z}}$ has Zeno behavior to the right of any instant $k\delta$.

Through the usual case analysis on the structure of formulas, we can prove that the behavior of any MTL formula over b_δ and $b_\delta^{\mathcal{Z}}$ is very simple, as it coincides with one of $\mathsf{p}, \neg\mathsf{p}, \top, \bot$ (see [17] for all details).

Lemma 4. *The truth value of any* MTL *formulas ϕ of granularity δ coincides with one of $\mathsf{p}, \neg\mathsf{p}, \top, \bot$ over both b_δ and $b_\delta^{\mathcal{Z}}$.*

An immediate consequence of the previous lemma is that, at any instant where the values $b_\delta(t)$ and $b_\delta^{\mathcal{Z}}(t)$ coincide, the truth values of any formula ϕ also coincide.

Corollary 1. *For any* MTL *formula ϕ of granularity δ, and all $k \in \mathbb{Z}$: $b_\delta(k\delta) \models_{\mathbb{R}} \phi$ iff $b_\delta^{\mathcal{Z}}(k\delta) \models_{\mathbb{R}} \phi$.*

Finally, we prove the desired separation result as follows.

Theorem 7. *If Zeno behaviors are allowed,* MTL $\subset \widetilde{\text{MTL}}$.

Proof. Let us consider the two families of behaviors: $\mathcal{N} = \{b_\delta \mid \delta \in \mathbb{Q}_{>0}\}$ and $\mathcal{Z} = \{b_\delta^{\mathcal{Z}} \mid \delta \in \mathbb{Q}_{>0}\}$.

First, let us consider initial satisfiability. The $\widetilde{\text{MTL}}$ formula $\Sigma = \widetilde{\bigcirc}(\neg\mathsf{p})$ separates initially the two families $\mathcal{N}$ and $\mathcal{Z}$, as $b(0) \models_{\mathbb{R}} \Sigma$ for all $b \in \mathcal{N}$ and $b'(0) \not\models_{\mathbb{R}} \Sigma$ for all $b' \in \mathcal{Z}$.

On the contrary, let ϕ be any MTL formula, and let δ be its granularity. Then, $\mathcal{N} \ni b_\delta(0) \models_{\mathbb{R}} \phi$ iff $\mathcal{Z} \ni b_\delta^{\mathcal{Z}}(0) \models_{\mathbb{R}} \phi$ by Corollary 1, so no MTL formula separates initially the two families. This implies that the $\widetilde{\text{MTL}}$ formula Σ has no initially equivalent formula in MTL.

Now, let us consider global satisfiability. The $\widetilde{\text{MTL}}$ formula $\Sigma' = \mathsf{p} \Rightarrow \widetilde{\bigcirc}(\neg\mathsf{p})$ separates globally the two families $\mathcal{N}$ and $\mathcal{Z}$, as $b(t) \models_{\mathbb{R}} \Sigma'$ for all $t \in \mathbb{T}$ and for all $b \in \mathcal{N}$, and $b'(t) \not\models_{\mathbb{R}} \Sigma'$ for some $t = k\delta$, and for all $b' \in \mathcal{Z}$.

On the contrary, let ϕ be any MTL formula, and let δ be its granularity. For the sake of contradiction, assume that $b(t) \models_{\mathbb{R}} \phi$ for all $t \in \mathbb{T}$ and for all $b \in \mathcal{N}$, and that $b'(t) \not\models_{\mathbb{R}} \phi$ for some t, and for all $b' \in \mathcal{Z}$. Now, in particular,

$b_\delta \models_{\mathbb{R}} \phi$; *a fortiori*, $b_\delta(0) \models_{\mathbb{R}} \phi'$ where $\phi' = \Box_{[0,+\infty)}(\phi)$. Similarly, it must be $b_\delta^{\mathcal{Z}} \not\models_{\mathbb{R}} \phi$. A little reasoning should convince us that this implies $b_\delta^{\mathcal{Z}}(0) \not\models_{\mathbb{R}} \phi'$. In fact, $b_\delta^{\mathcal{Z}} \not\models_{\mathbb{R}} \phi$ means that there exists a $t \in \mathbb{T}$ such that $b_\delta^{\mathcal{Z}}(t) \not\models_{\mathbb{R}} \phi$. t may be greater than, equal to, or less than 0. However, $b_\delta^{\mathcal{Z}}$ is *periodic* with period δ; this implies that $b_\delta^{\mathcal{Z}}(t) \models_{\mathbb{R}} \alpha$ iff $b_\delta^{\mathcal{Z}}(t + k\delta) \models_{\mathbb{R}} \alpha$, for all formulas α, $t \in \mathbb{T}$, $k \in \mathbb{Z}$. Therefore, if there exists a $t \in \mathbb{T}$ such that $b_\delta^{\mathcal{Z}}(t) \not\models_{\mathbb{R}} \phi$, then also there exists a $t' \geq 0$ such that $b_\delta^{\mathcal{Z}}(t') \not\models_{\mathbb{R}} \phi$. The last formula implies that $b_\delta^{\mathcal{Z}}(0) \not\models_{\mathbb{R}} \phi'$.

Now, notice that the formula ϕ' is of the same granularity as ϕ, that is δ. Moreover, $b_\delta(0) \models_{\mathbb{R}} \phi'$ and $b_\delta^{\mathcal{Z}}(0) \not\models_{\mathbb{R}} \phi'$. This contradicts Corollary 1; therefore ϕ does not globally separate the two families of behaviors. Since ϕ is generic, the $\widetilde{\text{MTL}}$ formula Σ' has no globally equivalent formula in MTL. □

Finally, if we reconsider all the theorems of the current section, and the corresponding proofs, we notice that they still stand for the matching variants of the non-strict and strict *until*. In other words, the same proofs provide a separation between $\text{MTL}^{\downarrow}$ and $\widetilde{\text{MTL}}^{\downarrow}$.

5.2 Matching as Expressive as Non-Matching

A careful reconsideration of the proofs of (6–9) shows that the equivalences hold even when Zeno behaviors are allowed; essentially, Zeno behaviors can be explicitly dealt with in the proof.[4] In summary, we have a proof that $\widetilde{\text{MTL}} = \widetilde{\text{MTL}}^{\downarrow}$ even if Zeno behaviors are allowed. As an example, let us sketch a bit of the proof of (6) for Zeno behaviors.

Proof (of (6) for Zeno behaviors). For the $\Leftarrow$ direction, let us consider the case: $b(t) \models_{\mathbb{R}} \neg\widetilde{\bigcirc}(\phi_2 \wedge \phi_1)$ and $b(t) \models_{\mathbb{R}} \widetilde{\bigcirc}(\phi_1)$, i.e., ϕ_1 holds over an interval $(t, t+\epsilon)$ for some $\epsilon > 0$. If ϕ_2 has Zeno behavior to the right of t, it changes truth value infinitely many times over $\min(\epsilon, b)$. Hence, there exists a $0 < \nu < \min(\epsilon, b)$ such that $b(t + \nu) \models_{\mathbb{R}} \phi_2$; so $b(t) \models_{\mathbb{R}} \widetilde{\mathsf{U}}_{\langle 0,b\rangle}(\phi_1, \phi_2)$ *a fortiori*. The other cases are done similarly (see [17]). □

Furthermore, it is possible to adapt (6–9) to use non-strict operators only. In practice, (7–9) hold if we just replace strict operators with the corresponding non-strict ones; on the other hand, (6) should be modified as:

$$\mathsf{U}_{\langle 0,b\rangle}(\phi_1, \phi_2) \equiv \mathsf{U}^{\downarrow}_{\langle 0,b\rangle}(\phi_1, \phi_2) \vee (\Diamond_{\langle 0,b\rangle}(\phi_2) \wedge \bigcirc(\phi_1) \wedge \mathsf{R}^{\downarrow}_{\langle 0,b\rangle}(\phi_2 \wedge \neg\phi_1, \bigcirc(\phi_1)) \tag{10}$$

All the resulting new equivalences using only non-strict operators can be shown to hold for Zeno, as well as non-Zeno, behaviors (see [17]). Hence, we have a proof that $\text{MTL} = \text{MTL}^{\downarrow}$ over generic behaviors.

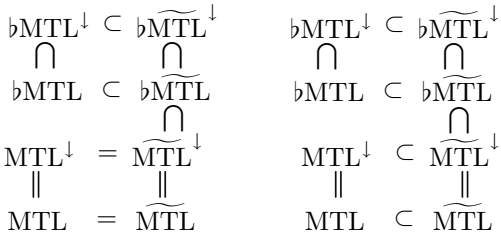

Fig. 2. Expressiveness over non-Zeno (left) and Zeno (right) behaviors.

6 Summary and Discussion

Figure 2 displays the relative expressiveness relations for non-Zeno behaviors (left) and Zeno behaviors (right). Note that the separation proofs for the flat fragments used only non-Zeno behaviors, therefore they imply the separation of the corresponding classes for generic (i.e., including Zeno) behaviors as well. On the other hand, the problem of the relative expressiveness of ♭MTL and $\flat\widetilde{\mathrm{MTL}}^{\downarrow}$ is currently open (over both Zeno and non-Zeno behaviors).

Tackling this open question about MTL relative expressiveness, and considering other variations such as the use of past operators, belongs to future work.

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[4] We are grateful to anonymous referee #4, whose detailed comments prompted us to realize this fact.

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